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Brouwer degree, domination of manifolds, and groups presentable by products*

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Abstract. For oriented connected closed manifolds of the same dimension, there is a transitive relation: M *dominates* N, or $M \geq N$, if there exists a continuous map of non-zero degree from M onto N. Section 1 is a reminder on the notion of degree (Brouwer, Hopf), Section 2 shows examples of domination and a first set of obstructions to domination due to Hopf, and Section 3 describes obstructions in terms of Gromov's simplicial volume.

In Section 4 we address the particular question of when a given manifold can (or cannot) be dominated by a product. These considerations suggest a notion for groups (fundamental groups), due to D. Kotschick and C. Löh: a group is *presentable by a product* if it contains two infinite commuting subgroups which generate a subgroup of finite index. The last section shows a small sample of groups which are not presentable by products; examples include appropriate Coxeter groups.

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1. Two definitions of the Brouwer degree

Degrees as defined below can be traced back to Brouwer [Brou–11]. Standard references for the Brouwer degree include [Pont-59, § 9], [Miln–65], [Hirs–76], and [Pras–06, § 5.5].

Consider a fixed dimension $n \geq 1$ and two oriented connected closed smooth manifolds M, N of dimension n. Let $f : M \longrightarrow N$ be a smooth map. Recall that a *regular value* of f is a point $y \in N$ such that the differential of f is invertible at each point of $f^{-1}(y)$. It is a basic result due to A. Sard (1942) that the complement in N of the set of regular values has Lebesgue measure zero; in particular, the set of regular values of f is dense [Brow–35, Theorem 3-III], and a fortiori non-empty. For more recent proofs of the results of Brown and Sard, see e.g. [Rham–60, Page 10] or [Miln–65, § 2 & 3].

If $y \in N$ is a regular value of f, then f is a local diffeomorphism around every $x \in f^{-1}(y)$; it follows that $f^{-1}(y)$ is discrete in M; since $f^{-1}(y)$ is also compact, it is therefore finite. For $x \in f^{-1}(y)$ define $\varepsilon_x(f)$ to be 1 if f is orientation preserving at x and -1 if f is orientation reversing at x.

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Definition 1.1. The **local degree of f at a regular value** y is the integer

$$(\Sigma) \qquad \deg_{(y)}(f) \, = \sum_{x \in f^{-1}(y)} \varepsilon_x(f).$$

It is a particular case of the definition that $\deg_{(y)}(f) = 0$ if f is *not* surjective and y not in the image of M.

Proposition 1.2. Let $f : M \longrightarrow N$ be a smooth map and y a regular value of f, as above.

(1) The local degree $\deg_{(y)}(f)$ defined in (Σ) is independent on the choice of the regular value y.
(2) For another smooth map $f' : M \longrightarrow N$ homotopic to f and a regular value y' of f', the local degrees coincide: $\deg_{(y')}(f') = \deg_{(y)}(f)$.

Reminder 1.3 (fundamental class)**.** For a connected closed manifold M of dimension n, the homology group $H_n(M; \mathbb{Z})$ is either isomorphic to $\mathbb{Z}$, in which case M is **orientable**, or is $\{0\}$. When M is orientable, an **orientation** of M is a choice of generator

$$[M] \, \in \, H_n(M; \mathbb{Z}) \simeq \mathbb{Z},$$

called the **fundamental class** of M. See for example [Hatc–02, Chapter 3]; more on the fundamental class in 1.9 below.

Notation 1.4 (induced maps). Let X, Y be topological spaces, $f : X \longrightarrow Y$ a continuous map, j a non-negative integer, and A an abelian group. We denote by

$$f_j \, : \, H_j(X; A) \longrightarrow H_j(Y; A)$$

the map induced by f in homology with coefficients in A, and by

$$f^j \, : \, H^j(Y; A) \longrightarrow H^j(X; A)$$

the map induced by f in cohomology.

If X, Y are connected, and good enough to have a standard theory of fundamental groups and coverings (in particular if X and Y are connected manifolds), and if base points have been chosen, we denote by

$$f_\pi \, : \, \pi_1(X) \longrightarrow \pi_1(Y)$$

the homomorphism induced on fundamental groups.

Proof of Proposition 1.2. Let D_y be a small embedded open disc (or simplex) in N centred at y. If D_y is small enough, there exists for all $x \in f^{-1}(y)$ an open disc D_x such that the restriction of f to D_x is a diffeomorphism, say g_x, from D_x onto D_y, and the D_x 's are disjoint. Observe that g_x is orientation preserving [respectively orientation reversing] when $\varepsilon_x(f) = 1$ [resp. when $\varepsilon_x(f) = -1$].

Let S denote an oriented sphere of dimension n, with a base point p. Let $\pi_N : N \longrightarrow S$ be an orientation-preserving map obtained by mapping D_y diffeomorphically onto the complement of p in S, and collapsing the complement of D_y in N to p. Let $\pi_M : M \longrightarrow S$ be the map which coincides with $\pi_N \circ g_x$ on D_x, for each $x \in f^{-1}(y)$, and collapses the complement of $\bigsqcup_{x \in f^{-1}(y)} D_x$ in M to p. The diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ \downarrow{\scriptstyle \pi_M} & & \downarrow{\scriptstyle \pi_N} \\ S & \xrightarrow{\mathrm{id}} & S \end{array}$$

commutes (where id denotes the identity).

The corresponding maps on the top-dimensional homology groups provide the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[M] = H_n(M;\mathbb{Z}) & \xrightarrow{f_n} & H_n(N;\mathbb{Z}) = \mathbb{Z}[N] \\ \downarrow{\scriptstyle (\pi_M)_n} & & \downarrow{\scriptstyle (\pi_N)_n} \\ \mathbb{Z}[S] = H_n(S;\mathbb{Z}) & \xrightarrow{\mathrm{id}} & H_n(S;\mathbb{Z}) = \mathbb{Z}[S] \end{array} .$$

We have $(\pi_N)_n([N]) = [S]$ and $(\pi_M)_n([M]) = \sum_{x \in f^{-1}(y)} \varepsilon_x(f)[S] = \deg_{(y)}(f)[S]$. Hence

$$f_n([M]) \; = \; \deg_{(y)}(f)[N].$$

Since the left-hand side is independent of y, Claim (1) holds. Claim (2) follows because $f'_n = f_n$. □

As a consequence, we can now define the degree of f, denoted by $\deg(f)$, to be the number that appears in (Σ), for any regular value y of f. But, as a consequence of Proposition 1.2, more is true, and we can define the degree of any *continuous* map between manifolds.

Definition 1.5. Let M and N be two oriented connected closed topological manifolds of the same dimension, say $n \geq 1$, and $f : M \longrightarrow N$ a continuous map. The **degree** of f is the integer $\deg(f) \in \mathbb{Z}$ such that

$$\text{(defdeg)} \qquad f_n([M]) \; = \; \deg(f)[N] \; \in \; H_n(N;\mathbb{Z}).$$

This definition is due to Hopf: see Formula (3) in [Hopf–30].

In Definition 1.5, assume moreover that M and N are smooth. Claim: f can be approximated by smooth maps. For a precise definition of the appropriate topology on the space $\mathcal{C}(M,N)$ of continuous map from M to N and for a proof of the claim, see [Hirs–76], Chapter 2, in particular Theorem 2.6. Alternatively, see [Pras–06], Chapter 5, Section 4.4.

Since two smooth approximations g_0, g_1 of f are homotopic (being close to each other), their local degrees are equal, by Proposition 1.2. Hence the local degrees of Definition 1.1 coincide with the homologically defined degree of Definition 1.5.

As a digression, we note that this strategy applies to spaces of maps from M to N larger than the space of continuous maps; see [BrNi–95].

Brouwer's definition keeps its merits for many purposes, in particular to suggest definitions in a setting adapted to functional analysis and differential equations, as in a very influential 1934 paper by Leray and Schauder [LeSc–34].

Note also that the connectedness of M in Definition 1.5 is not an issue: in case M has several connected components M_i, the degree of $f : M \longrightarrow N$ is the sum of the degrees of the restrictions $f|_{M_i}$.

The two claims of the following proposition are straightforward consequences of Definition 1.5.

Proposition 1.6. Let M, N, P be oriented connected closed manifolds of the same dimension n.

(1) Let $f : M \longrightarrow N$ and $g : N \longrightarrow P$ be continuous maps. Then $\deg(g \circ f) = \deg(f)\deg(g)$.

(2) Suppose that $M = M_1 \times M_2$ and $N = N_1 \times N_2$ are product manifolds, where M_1, M_2, N_1, N_2 are oriented connected closed manifolds with $\dim(M_1) = \dim(M_2)$ and $\dim(N_1) = \dim(N_2)$, and that $f : M \longrightarrow N$ is a product of two continuous maps $f_j : M_j \longrightarrow N_j$ $(j = 1, 2)$. Then $\deg(f) = \deg(f_1)\deg(f_2)$.

Remark 1.7. Besides (1) and (2) in Proposition 1.2, the following important step in the standard proof of the proposition has an independent interest (see [Miln–65, § 5, Lemma 1]):

> *suppose that M is the boundary of an oriented connected smooth manifold with boundary, say X, of dimension $n + 1$, and that f extends to a smooth map $X \longrightarrow N$; then* $\deg(f) = 0$.

Indeed, f extends if and only if $\deg(f) = 0$ [Hirs–76, Chap. 5, Th. 1.8].

Remark 1.8 (history, homology and groups)**.** The formulation above is completely standard nowadays, and rather far from the original formulation of Brouwer, for several reasons.

One is that Brouwer used "piecewise linear" approximations (rather than smooth approximations) of continuous mappings.

Another one is that the notion of group was not used in homology theory before the mid 20's, when homology invariants were promoted from numbers (Riemann, Betti, Poincaré) to groups, under the strong influence of Emmy Noether and a few others (Hopf, Mayer, Vietoris). E. Noether gave lectures on the subject in Göttingen, but apparently did not publish anything but a 14-lines report[2] dated 27. Januar 1925 [Noet–25]; see [Hirz–96].

Heinz Hopf had received his doctorate in 1925 in Berlin, from Erhard Schmidt. He went then to Göttingen as a young "postdoc", and met E. Noether and P. Alexandroff. The book "Topologie" by Alexandroff and Hopf was published in 1935 [AlHo–35], with the dedication

L.E.J. Brouwer gewidmet.

Topology was not quite a respectable subject in these times; note for example that it plays a rather minor role in Hilbert's problems. It seems that Hilbert was not comfortable with Poincaré's highly intuitive approach to topology, involving "mistakes", corrections, and imprecise statements. From the middle 20's onwards, topology became somehow respectable [Eckm–06]; this owes a lot to the work of Hopf, and a few others including Alexander and de Rham.

Thom gave his view on the Brouwer degree in modern differential topology [Thom–71].

[2]Noether's report is about the structure of finitely generated abelian groups. In its 14 lines, topology is mentioned in the last phrase only: "Der Gruppensatz erweist sich so als der einfachere Satz; in den Anwendungen des Gruppensatzes – z.B. Bettische und Torsionszahlen in der Topolgie – ist somit ein Zurückgehen auf die Elementarteilertheorie nicht erforderlich."

Remark 1.9 (dual fundamental classes)**.** Let M, N are orientable connected closed manifolds of the same dimension, $n \geq 1$, and $f : M \longrightarrow N$ a continuous map.

It follows from the Universal Coefficient Theorem for cohomology that we have an isomorphism $H^n(M;\mathbb{Z}) \simeq \mathrm{Hom}(H_n(M;\mathbb{Z}),\mathbb{Z})$, and therefore that $\mathrm{Hom}(H_n(M;\mathbb{Z}),\mathbb{Z})$ is isomorphic to $\mathbb{Z}$. The **dual fundamental class** is the element $[M]^* \in H^n(M;\mathbb{Z})$ which maps $[M] \in H_n(M;\mathbb{Z})$ to $1 \in \mathbb{Z}$. Similarly for $[N]^* \in H^n(M;\mathbb{Z})$.
The induced map on $H^n(\cdot;\mathbb{Z})$ is the transposed

$$f^n \ : \mathrm{Hom}(H_n(N;\mathbb{Z}),\mathbb{Z}) \longrightarrow \mathrm{Hom}(H_n(M;\mathbb{Z}),\mathbb{Z}), \ \ c \longmapsto c \circ f_n$$

of the map $f_n : H_n(M;\mathbb{Z}) \longrightarrow H_n(N;\mathbb{Z})$ in homology. Hence we have a dual formulation

$$\text{(defdegco)} \qquad (f)^n([N]^*) \ = \ \deg(f)[M]^* \ \in \ H^n(M;\mathbb{Z})$$

of the definition via (defdeg) in 1.5.

A particular case of the Universal Coefficient Theorem for homology establishes that the natural map $H_n(M;\mathbb{Z}) \otimes \mathbb{Q} \longrightarrow H_n(M;\mathbb{Q})$ is an isomorphism. The image of $[M] \otimes 1$ is not zero, and is consequently a basis of the one-dimensional $\mathbb{Q}$-vector space $H_n(M;\mathbb{Q})$; it is denoted by $[M]$ again. Similarly for $[N] \in H_n(N;\mathbb{Q})$.
For a map g of M to itself, the degree appears as the multiplication factor of the induced endomorphism of $H_n(M;\mathbb{Q})$, without reference to $H_n(M;\mathbb{Z})$; but this conceals that $\deg(g)$ is always an *integer*. For $f : M \longrightarrow N$, the degree of f is again given by $f_n([M]) = \deg(f)[N] \in H_n(N;\mathbb{Q})$; but there is in this case a reference to $H_n(\cdot;\mathbb{Z})$, because $H_n(M;\mathbb{Q})$ and $H_n(N;\mathbb{Q})$ *have to* be viewed together with distinguished elements, which are images of fundamental classes defined in integral homology.

Examples 1.10. Let M, N be as in Definition 1.5.

(1) For every $d \in \mathbb{Z}$, there exists a continuous map $M \longrightarrow S^n$ of degree d.
This holds if M is the circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$, because the map $z \longmapsto z^d$ of the circle into itself is of degree d. This holds if M is a sphere of larger dimension n, by induction: indeed, if φ is a continuous map of a n-sphere into itself, its suspension is a continuous map of a $(n+1)$-sphere into itself of the same degree as φ. Consider now the general case. There exists a map from M to S^n of degree 1 (see the proof of Proposition 1.2); the composition with a map from S^n to itself of degree d is a map from M to S^n of degree d.
It is a famous theorem of Hopf [Hopf–27] that:

two continuous maps from M to the n-sphere are homotopic
if and only if their degrees coincide.

Indeed, Whitney proved later (1937) the following generalization: if X is any connected polyhedron of dimension at most n, homotopy classes of continuous maps $f : X \longrightarrow S^n$ are in bijection with $H^n(X;\mathbb{Z})$, the homotopy class of f corresponding to the image $f^n([S^n]^*) \in H^n(X;\mathbb{Z})$, where $[S^n]^*$ is the dual fundamental class of the n-sphere.

(2) If there exists a homeomorphism f from M onto N, then $\deg(f) = 1$ if f is orientation-preserving, and $\deg(f) = -1$ if f is orientation-reversing. The conclusion carries over to the case of a homotopy equivalence $f : M \longrightarrow N$.
There are known examples of pairs (M, N) of non-homeomorphic manifolds for which there exists a homotopy equivalence from M to N. In dimension 3, there is a standard example, that of the pair of lens spaces $L(7,1)$, $L(7,2)$.

Reminder on lens spaces. For $p \geq 2$ and $k \in \{1, \dots, p\}$ coprime with p, let $L(p, k)$ denote the quotient of the 3-sphere $\{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$ by an action of the cyclic group $\mathbb{Z}/p\mathbb{Z}$ for which a generator acts by $(z_1, z_2) \longmapsto (\exp(2\pi i/p)z_1, \exp(2\pi ik/p)z_2)$. We have $\pi_1(L_p(p, k)) \simeq \mathbb{Z}/p\mathbb{Z}$.

In the mid 30's, Reidemeister has found a necessary condition for $L(p, k)$ and $L(p, \ell)$ to be PL-homeomorphic; in particular $L(7, 1)$ and $L(7, 2)$ are NOT homeomorphic. Rueff, and slightly later Franz, have studied conditions for $L(p, k)$ and $L(p, \ell)$ to have the same homotopy type; in particular $L(7, 1)$ and $L(7, 2)$ DO have the same homotopy type. More on lens spaces in [Rham–67, Miln–66, Volk–13].

The following result uses a lot, including Perelman's work: Let M, N be two irreducible connected closed 3-manifold which have the same homotopy type. At least one of the following is true: (i) M and N are homeomorphic, (ii) M and N are lens spaces. [From a conversation with Claude Weber.]

(3) If there exists a covering $f : M \longrightarrow N$ with k leaves, then $\deg(f) = k$ or $\deg(f) = -k$.

This carries over to branched coverings, i.e. to maps $f : M \longrightarrow N$ for which there is a subset $B \subset M$ "small enough" (e.g. of codimension at least two) such that the restriction of f to $M \smallsetminus B$ is a covering over $N \smallsetminus f(B)$.

(4) Suppose that M is a connected sum $M = N \sharp P$ of two oriented connected closed manifolds N, P of the same dimension. The manifold obtained from M by collapsing P to a point can be identified with N, so that there is a map $f : M \longrightarrow N$ well defined up to homotopy. Then $\deg(f) = 1$.

In particular, since M and $M \sharp S^n$ are homeomorphic, we obtain again a map $M \longrightarrow S^n$ of degree 1, as in (1).

(5) Suppose that Σ_g and Σ_h are oriented connected closed surfaces, of genus g and h respectively. The following properties are equivalent:

- $g \geq h$,
- there exists a continuous map f from Σ_g to Σ_h of non-zero degree,
- there exists a continuous map f from Σ_g to Σ_h of degree 1.

The equalities in (2), (3) and (4) follow from the definitions, and the equivalences of (5) are left as an exercise for the reader. [Hint: to exclude the existence of maps with non-zero degrees, use Proposition 2.6(2) below; see Example 2.8(3).]

(6) For every continuous map f of the complex projective space $P^n_{\mathbb{C}}$ to itself, the degree is an integer of the form d^n, for some $d \in \mathbb{Z}$.

Indeed, on the one hand, if h denotes the standard generator of $H^2(P^n_{\mathbb{C}}; \mathbb{Z}) \simeq \mathbb{Z}$, so that h^n is the dual fundamental class $[P^n_{\mathbb{C}}]^* \in H^{2n}(P^n_{\mathbb{C}}; \mathbb{Z})$, and if d is the integer such that $f^2(h) = dh$, then $f^{2n}([P^n_{\mathbb{C}}]^*) = (f^2(h))^n = (dh)^n = d^n[P^n_{\mathbb{C}}]^*$.

On the other hand, the map given in homogeneous coordinates by

$$P^n_{\mathbb{C}} \longrightarrow P^n_{\mathbb{C}}, \ \ (z_0 : z_1 : \cdots : z_n) \longmapsto (z_0^d : z_1^d : \cdots : z_n^d)$$

has degree d^n for every $d \in \mathbb{N}$, and that given by $(z_0 : z_1 : \cdots : z_n) \longmapsto (\overline{z_0} : \overline{z_1} : \cdots : \overline{z_n})$ has degree $(-1)^n$.

There are many important statements with simple proofs involving the notion of Brouwer degree. Here is a standard sample.

Brouwer Fixed Point Theorem 1.11. Every continuous mapping f from the n-disc D^n into itself has a fixed point.

Proof. We identify D^n to the upper hemisphere, defined by $x_n \geq 0$, of

$$S^n = \Big\{ (x_0, \dots, x_n) \in \mathbb{R}^n \ \Big| \ \sum_{i=0}^{n} x_i^2 = 1 \Big\}.$$

Define a transformation g of S^n by

$$g(x) = \begin{cases} f(x) & \text{if } x \in D^n \subset S^n \\ f(x_0, \dots, x_{n-1}, -x_n) & \text{if } x = (x_0, \dots, x_n) \notin D^n, \ \ x \in S^n. \end{cases}$$

Then $\deg(g) = 0$, because g is not surjective.

Every fixed-point free transformation h of S^n has degree $(-1)^{n+1}$, because h is then homotopic to the antipodal map $a : x \longmapsto -x$, of degree $(-1)^{n+1}$; indeed, $H(t,x) = \frac{th(x)-(1-t)x}{\|th(x)-(1-t)x\|}$ defines a homotopy from $H(1,\cdot) = h$ to $H(0,\cdot) = a$, so that $\deg(h) = \deg(a) = (-1)^{n+1} \neq 0$.

Hence g has a fixed point, and every fixed point of g is also a fixed point of f. □

Brouwer Invariance of Domain Theorem 1.12. Let U be an open subset of $\mathbb{R}^n$ and $\varphi : U \longrightarrow \mathbb{R}^n$ a continuous injective map; then $f(U)$ is open.

For a proof using the notion of Brouwer degree, see e.g. [Tao–14, § 6.2].

Remark 1.13 (other manifolds)**.** In Definition 1.5, the requirements about M, N being closed can be relaxed.

(1) Consider manifolds with boundaries. Let M be a connected compact n-manifold, possibly with non-empty boundary ∂M. What has been recalled in 1.3 extends as follows: the homology group $H_n(M, \partial M; \mathbb{Z})$ is either isomorphic to $\mathbb{Z}$, in which case M is **orientable**, or is $\{0\}$. When M is orientable, an **orientation** of M is a choice of generator $[M, \partial M]$ in $H_n(M, \partial M; \mathbb{Z}) \simeq \mathbb{Z}$, and $[M, \partial M]$ is then called the **fundamental class** of M.

Let M, N be two such oriented manifolds. The degree of a continuous map f from M to N such that $f(\partial M) \subset \partial N$ is the integer d such that

$$(\sharp) \qquad f_n([M, \partial M]) = d[N, \partial N],$$

where $f_n : H_n(M, \partial M; \mathbb{Z}) \longrightarrow H_n(N, \partial N; \mathbb{Z})$ is induced by f.

Supposer moreover that M and N are smooth. A map $f : (M, \partial M) \longrightarrow (N, \partial N)$ as above is homotopic to a smooth map, say g. Such a smooth map has regular values in $N \smallsetminus \partial N$, and the degree $\deg(f) = \deg(g)$ can be computed as in Equation (Σ) of Definition 1.1.

(2) Let now M, N be two orientable connected n-manifolds without boundary (note that M, N need not be compact). If M, N are smooth, Definition 1.1 and Proposition 1.2 carry over to *proper* smooth maps from M to N.

If M is not compact, recall that $H_n(M; \mathbb{Z}) = \{0\}$. But Definition 1.5 can be adapted by using an appropriate notion of homology for which an orientable manifold need not be compact to have a fundamental class, for example by using the locally finite homology exposed in [Geog–08, Part III].

2. Domination

Let M, N be oriented connected closed manifolds of dimension n.

Definition 2.1. The manifold M **dominates** N if there exists a continuous map of non-zero degree from M to N. More precisely, for $d \geq 1$, the manifold M d-dominates N if there exists a continous map of degree $\pm d$ from M to N.

Watch out! Let X, Y be topological spaces. Then X *dominates* Y in the sense of J.H.C. Whitehead if there exist continuous maps $i : Y \longrightarrow X$ and $r : X \longrightarrow Y$ such that $r \circ i$ is homotopic to id_X. If X dominates Y in this sense, then $r_j : H_j(X; \mathbb{Z}) \longrightarrow H_j(Y; \mathbb{Z})$ is surjective for all $j \geq 0$. See [Whad–48] and [Hatc–02, Appendix].

Below, domination will always refer to Definition 2.1, and not to Whitehead's notion.

The following structuring question has been asked by Gromov in a talk from 1978 (according to [CaTo–89]). It is of course present, implicitly but strongly, in [Hopf–30], and also later in [MiTh–77] and [Grom–82].

Question 2.2. Let M, N be as above. Can one decide whether or not M dominates N ?

Examples 2.3. Each of (1) to (5) below is related to the same number in Example 1.10 above.

(1) Every n-manifold M as above dominates the n-sphere. Compare with the proof of Proposition 1.2, and with (4) in Example 1.10.

(2) Two manifolds of the same homotopy type dominate each other.

(3) A covering or a branched covering with a finite number of sheets is a domination. For a specific example, consider a non-trivial finite subgroup Γ of the 3-sphere S^3, identified to the special unitary group $\mathrm{SU}(2)$: set $N = S^3/\Gamma$ and $d = |\Gamma|$. There exists a covering $S^3 \longrightarrow N$ of degree d (or $-d$ for "the other" orientation of N) and a smooth map $N \longrightarrow S^3$ of degree 1 (as in (1) above). Hence each of N, S^3 dominates the other.

(4) A connected sum of two oriented manifolds dominates each of them.

(5) In particular, an oriented connected closed surface dominates every such surface of lower genus.

(6) For every oriented connected closed 3-manifold N, there exists a *hyperbolic* 3-manifold M which dominates N. This is a consequence of results of Sakuma and Brooks [Broo–85], revisited in [Mont–87].
Theorem 2.13 for $n = 3$ gives a stronger result.

(7) For $r \geq 1$, let $P^r_{\mathbb{C}}$ denote the complex projective space of complex dimension r. Consider two integers $p, q \geq 1$ and their sum $n = p + q$. The product $P^p_{\mathbb{C}} \times P^q_{\mathbb{C}}$ dominates $P^n_{\mathbb{C}}$.
Here is one way to construct a holomorphic map $P^p_{\mathbb{C}} \times P^q_{\mathbb{C}} \longrightarrow P^n_{\mathbb{C}}$ of Brouwer degree $\binom{n}{p}$. The starting ingredient is the Segre embedding

$$\sigma_{[(p+1)(q+1)-1]} \ : \ \left\{ \begin{array}{ccc} P^p_{\mathbb{C}} \times P^q_{\mathbb{C}} & \longrightarrow & P^{(p+1)(q+1)-1}_{\mathbb{C}} \\ \big((u_0 : u_1 : \cdots : u_p), (v_0 : v_1 : \cdots : v_q)\big) & \longrightarrow & (\cdots : u_i v_j : \cdots) \end{array} \right.$$

(the notation $(\cdots : \cdots : \cdots)$ indicates homogeneous coordinates).
For k with $(p+1)(q+1) - 1 > k \geq n$, define inductively a mapping $\sigma_{[k]}$ from the image of $\sigma_{[k+1]}$ to $P^k_{\mathbb{C}}$ as follows: choose a point $a \in P^{k+1}_{\mathbb{C}}$ not in the image of $\sigma_{[k+1]}$, and define $\sigma_{[k]}(x)$ as the intersection with $P^k_{\mathbb{C}}$ (identified to a hyper surface of $P^{k+1}_{\mathbb{C}}$)

of the line containing a and x, for all x in the image of $\sigma_{[k+1]}$. It can be checked that the composition

$$\sigma \;:=\; \sigma_{[n]} \circ \sigma_{[n+1]} \circ \cdots \circ \sigma_{[(p+1)(q+1)-2]} \circ \sigma_{[(p+1)(q+1)-1]} \;:\; P^p_{\mathbb{C}} \times P^q_{\mathbb{C}} \longrightarrow P^n_{\mathbb{C}}$$

is a map of degree $\binom{n}{p}$.

Let us consider the special case $P^1_{\mathbb{C}} \times P^2_{\mathbb{C}} \longrightarrow P^3_{\mathbb{C}}$. There are rational maps

$$\begin{aligned}
\sigma_{[5]} &: P^1_{\mathbb{C}} \times P^2_{\mathbb{C}} \longrightarrow P^5_{\mathbb{C}}, \quad \big((u:v),(x:y:z)\big) \longmapsto (ux:uy:uz:vx:vy:vz),\\
\rho^5_4 &: P^5_{\mathbb{C}} \dashrightarrow P^4_{\mathbb{C}}, \quad (x_0:x_1:x_2:x_3:x_4:x_5) \mapsto (x_1:x_2:x_3:x_4:x_5-x_0),\\
\rho^4_3 &: P^4_{\mathbb{C}} \dashrightarrow P^3_{\mathbb{C}}, \quad (y_0:y_1:y_2:y_3:y_4) \mapsto (y_1:y_2-y_o:y_3:y_4).
\end{aligned}$$

Observe that

- ρ^5_4 is not defined in $(1:0:0:0:0:1)$, which is a point outside Image$(\sigma_{[5]})$,
- $\rho^5_4 \circ \sigma_{[5]} : \big((u:v),(x:y:z)\big) \longmapsto (uy:uz:vx:vy:vz-ux)$ is defined on the whole of $P^1_{\mathbb{C}} \times P^2_{\mathbb{C}}$,
- ρ^4_3 is not defined in $(1:0:1:0:0)$, which is a point outside Image$(\rho^5_4 \circ \sigma_{[5]})$,

so that

$$\sigma := \rho^4_3 \circ \rho^5_4 \circ \sigma_{[5]} \;:\; \begin{cases} P^1_{\mathbb{C}} \times P^2_{\mathbb{C}} \longrightarrow P^3_{\mathbb{C}} \\ \big((u:v),(x:y:z)\big) \longmapsto (uz:vx-uy:vy:vz-ux) \end{cases}$$

is indeed defined everywhere. It can be checked that σ is of degree 3, using Definition (Σ) of 1.1 and the regular value $(1:1:1:1) \in P^3_{\mathbb{C}}$.

I am grateful to J.C. Hausmann and S. Zimmermann for discussions concerning this example.

Propositions 2.6 and 3.3 will establish *obstructions* to the domination of one manifold by another.

Before Proposition 2.6, let us recall one definition of Hopf's Umkehrungshomomorphismus [Hopf–30].

Definition 2.4. Let A be an abelian group. Let M, N be oriented connected closed manifolds of the same dimension. For $j \in \{1, \ldots, n\}$, let

$$P_j(M) \;:\; H^{n-j}(M;A) \longrightarrow H_j(M;A), \;\; c \longmapsto [M] \cap c$$

denote the Poincaré duality isomorphism, where $\cap$ stands for the cap product. Let $f : M \longrightarrow N$ be a continuous map.

The **Umkehrungshomomorphismus** $f_!$ is defined as the composition that makes the diagram

$$\text{(Umkehr)} \qquad \begin{array}{ccc}
H^{n-j}(M;A) & \xrightarrow[\approx]{P_j(M)} & H_j(M;A) \\
\uparrow f^{n-j} & & \uparrow f_! \\
H^{n-j}(N;A) & \xleftarrow[\approx]{P_j(N)^{-1}} & H_j(N;A)
\end{array}$$

commute. Note that $f_!$ is a "wrong way map" in homology; note also that neither f_* nor $f_!$ need respect the intersection product in homology.

Remark 2.5. (i) The previous definition is highly anachronic, using cohomology, as Freudenthal and others did later [Freu–37]. Indeed, cohomology with its product appeared in 1935 only, in communications by Alexander and Kolmogorov at the "Première Conférence internationale de Topologie", in Moscow, from 4th to 10th of September, 1935 [Mosc–35, Whey–88].

In 1930, Hopf could only use homology. We refer to [Hilt–88] for historical comments, and to [Geig–08, Section 3.4.1] for an exposition of the Umkehrungshomomorphismus in homology.

(ii) Despite (i), we like to think that the following proposition is due to Hopf. For (1) and (2) below, see respectively (B) on Page 86 and Satz Ib on Page 77 in [Hopf–30].

(iii) Digression: similarly, it was before the discovery of cohomology, and more precisely in 1931, that de Rham established a conjecture of E. Cartan and showed that the dimension of the "de Rham cohomology" $H^j_{\mathrm{dR}}(M;\mathbb{R})$ of a closed n-manifold M coincides with its jth Betti number.

Proposition 2.6 (Hopf)**.** Let M, N be oriented connected closed manifolds of the same dimension, n. Assume that there exists a continuous map $f : M \longrightarrow N$ of degree $d := \deg(f) \neq 0$, i.e. that M dominates N.

(1) The image of the homomorphism
$$f_\pi \, : \, \pi_1(M) \longrightarrow \pi_1(N)$$
is a subgroup of $\pi_1(N)$ of finite index, say c, and c divides $|d|$.

(2) For each $j \in \{0, 1, \dots, n\}$, the map
$$f_j \, : \, H_j(M;\mathbb{Q}) \longrightarrow H_j(N;\mathbb{Q})$$
in homology is surjective. In particular, there is an inequality
$$\dim(H_j(M;\mathbb{Q})) \, \geq \, \dim(H_j(N;\mathbb{Q}))$$
for the Betti numbers of M and N.

(3) For each $j \in \{0, 1, \dots, n\}$, the map
$$f^j \, : \, H^j(N;\mathbb{Q}) \longrightarrow H^j(M;\mathbb{Q})$$
in cohomology is injective.

(4) More precisely, the composition $f_* \circ f_! : H_*(N;\mathbb{Q}) \longrightarrow H_*(N;\mathbb{Q})$ is multiplication by the number d.

Note. We could equally use $\mathbb{R}$ instead of $\mathbb{Q}$ as coefficients.

Concerning (3), we will see below examples where it is important that f^* embeds the cohomology *algebra* $H^*(N;\mathbb{Q})$ as a subalgebra of $H^*(M;\mathbb{Q})$. In contrast, the induced maps of (2) *need not* preserve the intersection product (think of a map $S^1 \times S^1 \longrightarrow S^2$ of degree 1).

Proof. (1) Let Δ be the image of the induced homomorphism f_π from $\pi_1(M)$ to $\pi_1(N)$, and let $p : \widetilde{N} \longrightarrow N$ the covering of N corresponding to Δ. The map f factorizes as the composition of a lift $\widetilde{f} : M \longrightarrow \widetilde{N}$ of f with p.

If Δ were of infinite index in $\pi_1(N)$, the manifold $\widetilde{N}$ would be non-compact, the homology group $H_n(\widetilde{N};\mathbb{Z})$ the zero group, and $f_n = p_n \circ \widetilde{f}_n$ the zero map, contradicting $d \neq 0$. Hence Δ is of finite index, say of index c, in $\pi_1(N)$. Since the covering p is of degree c, the degree d of the composition $f = p \circ \widetilde{f}$ is the product of $\deg(\widetilde{f})$ and c.

(2)-(3)-(4) For $j \in \{0, 1, \dots, n\}$, consider the map F_j defined as that for which the diagram

$$\text{(F)} \qquad \begin{tikzcd} H^{n-j}(M;\mathbb{Q}) \arrow[r, "P_j(M)", "\approx"'] & H_j(M;\mathbb{Q}) \arrow[d, "f_j"] \\ H^{n-j}(N;\mathbb{Q}) \arrow[u, "f^{n-j}"] \arrow[r, dotted, "F_j"] & H_j(N;\mathbb{Q}) \end{tikzcd}$$

commutes. By naturality, F_j is the cup product

$$c \longmapsto f_n([M]) \cap c.$$

Since $f_n([M]) = d[N]$ with $d \neq 0$, the map F_j is a non-zero multiple of the Poincaré duality isomorphism $P_j(N)$, and in particular F_j is an isomorphism. It follows that f^{n-j} is injective and f_j surjective.

By comparing the diagrams (Umkehr) and (F), we see that $f_* \circ f_!$ is multiplication by d. □

Remark 2.7. Domination is often *thought of* as a **partial order** on the set $\mathcal{M}(n)$ of (homotopy types of) oriented connected closed manifolds of some dimension n, for which S^n is an absolutely minimal element. But it is not quite a partial order, as shown by S^3 and S^3/Γ as in Example 2.3(3).

Note that, in general, two manifolds in $\mathcal{M}(n)$ are not comparable for this "partial order", because neither of them dominates the other.

In the case of aspherical 3-manifolds, this order has been extensively studied by Wang [Wang–91]; see also [KoNe–13]. For non-hyperbolic 4-manifolds, see [Neof].

This "partial order" could be extended to pairs of oriented connected closed manifolds of possibly different dimensions, as suggested in [CaTo–89]: M **dominates** N if there exists a continuous map from M to N which is surjective in rational homology.

Examples 2.8. (1) If N is a manifold dominated by S^n, then $\pi_1(N)$ is finite, and N is a rational homology sphere, respectively by (1) and (3) in Proposition 2.6.

Observe that S^3 dominates infinitely many pairwise non-homeomorphic manifolds, quotients of S^3 by finite cyclic groups of all orders.

(2) The converse to a particular case of (1) holds as follows: *A simply connected rational homology n-sphere N is dominated by S^n.* This was shown to me by C. Neofytidis, with the following argument. For $j \in \{2, \dots, n-1\}$, the homology groups $H_j(N;\mathbb{Z})$ are in the Serre class $\mathcal{C}$ of finite abelian groups (see the discussion and the references in [Neof–14]). The Hurewicz theorem modulo the class $\mathcal{C}$ implies that the Hurewicz homomorphism $h : \pi_n(N) \longrightarrow H_n(N;\mathbb{Z})$ is an isomorphism modulo $\mathcal{C}$; in particular, the image of h is not $\{0\}$, and thus S^n dominates N.

(3) Let Σ_g and Σ_h be oriented connected closed surfaces, of genus g and h respectively, as in Example 1.10(5). If there exists a continuous map $\Sigma_g \longrightarrow \Sigma_h$ of non-zero degree, then $g \geq h$, by Proposition 2.6(2). This together with Example 2.3(5) shows that Σ_g dominates Σ_h if and only if $g \geq h$.

(4) None of the manifolds $S^2 \times S^4$ and $P^3_{\mathbb{C}}$ dominates the other. This follows from (3) in Proposition 2.6, but not from (2).

Indeed, the cohomologies

$$\begin{aligned} H^*(S^2 \times S^4; \mathbb{Q}) &= 1\mathbb{Q} \oplus c_2^S\mathbb{Q} \oplus c_4^S\mathbb{Q} \oplus [S^2 \times S^4]^*\mathbb{Q} \\ &\simeq \mathbb{Q}[x,y]/(x^2,y^2), \\ H^*(P^3_{\mathbb{C}}; \mathbb{Q}) &= 1\mathbb{Q} \oplus c_2^P\mathbb{Q} \oplus c_4^P\mathbb{Q} \oplus [P^3_{\mathbb{C}}]^*\mathbb{Q} \\ &\simeq \mathbb{Q}[z]/(z^4), \end{aligned}$$

are both generated, additively, by classes of degree $0, 2, 4, 6$. But neither of these algebras injects as a subalgebra in the other, since $(c_2^S)^2 = 0$ and $(c_2^P)^2 = c_4^P \neq 0$.

(5) Recall first that, if $M = M_1 \sharp M_2$ is a connected sum of two oriented connected manifolds, $H_j(M; \mathbb{Z}) \simeq H_j(M_1; \mathbb{Z}) \oplus H_j(M_2; \mathbb{Z})$ for all j with $1 \leq j \leq n-2$; if M_1 and M_2 are orientable, this holds also for $j = n-1$. The proof involves the long exact sequence for homology of the pair $(M_1 \sharp M_2, S^{n-1})$, showing that $H_j(M_1 \sharp M_2; \mathbb{Z})$ and $H_j(M_1 \vee M_2; \mathbb{Z})$ are isomorphic for $j \leq n-2$, and the Mayer-Vietoris exact sequence showing that $H_j(M_1 \vee M_2; \mathbb{Z}) \simeq H_j(M_1; \mathbb{Z}) \oplus H_j(M_2; \mathbb{Z})$ for all $j \geq 1$. Recall that the wedge sum $M_1 \vee M_2$ is the space obtained from the disjoint union $M_1 \sqcup M_2$ by identifying *one* point of M_1 with one of M_2.

Let M be the connected sum of two copies of $P^2_{\mathbb{C}}$ with the same orientation, so that the intersection form is definite. Let N be the connected sum of two copies of $P^2_{\mathbb{C}}$ with opposite orientations, so that the intersection form is indefinite. Then M and N have the same fundamental group (which is trivial) and the same additive homology and cohomology groups.

The manifold M does not dominate N, and N does not dominate M, because none of

$$\begin{aligned} H^*(M, \mathbb{Q}) &= H^*(P^2_{\mathbb{C}} \sharp P^2_{\mathbb{C}}; \mathbb{Q}) \simeq \mathbb{Q}[x,y]/(x^2 - y^2, x^3, y^3) \\ H^*(N, \mathbb{Q}) &= H^*(P^2_{\mathbb{C}} \sharp \overline{P^2_{\mathbb{C}}}; \mathbb{Q}) \simeq \mathbb{Q}[x,y]/(x^2 + y^2, x^3, y^3) \end{aligned}$$

is isomorphic to a subalgebra of the other.

I am grateful to D. Kotschick for this example. More generally, intersection forms play an important role for results and examples of domination for 4-manifolds [DuWa–04].

The following question has has attracted a lot of attention.

Question 2.9. Let n be an integer, $n \geq 3$. Let M be an orientable connected closed n-manifold, and $\mathcal{N}$ a class of such manifolds. Decide whether the number of N in $\mathcal{N}$ dominated by M is finite, up to homotopy equivalence, or up to homeomorphism when $n \leq 3$.

There is a natural variant of the question with 1-domination.

As a small sample of known answers to this kind of questions, we quote the following results.

Theorem 2.10. (1) For $n \geq 2$, every hyperbolic oriented connected closed n-manifold dominates a finite number only (up to homeomorphism) of such manifolds.

(2) Every oriented connected closed 3-manifold 1-dominates a finite number only (up to homeomorphism) of *geometrizable* oriented connected closed 3-manifolds.

(3) Every oriented connected closed 3-manifold dominates a finite number only (up to homeomorphism) of irreducible non-geometrizable oriented connected closed 3-manifolds.

Claim (1) has already been observed for $n = 2$ in Example 2.8(3). For $n \geq 4$, it follows from the fact that, for every $V > 0$, there are up to homeomorphism (indeed up to isometry) finitely many hyperbolic orientable connected closed manifolds of volume at most V (a particular case of [Wang–72, Theorem 8.1]), and from Gromov's work on simplicial volumes. For $n = 3$, we refer to [Soma–98]; note that [BoWa–96, Theorem 3.4] has been corrected [BoRW–14, Remark 1.5]. For (2) and (3), see respectively [Wang–02, Theorem 1.1] and [Liu, Theorem 1.1].

Theorem 2.11 (Boyer-Rolfsen-Wiest)**.** Let M, N be two oriented connected closed 3-manifolds. Suppose that M is prime, $\pi_1(N)$ left-orderable and non-trivial, and $\pi_1(M)$ not left-orderable.

Then M does not dominate N.

Recall that M is prime if, whenever M is a connected sum $M_1 \sharp M_2$, one of M_1, M_2 is a 3-sphere. By the Kneser-Milnor theorem, every oriented connected closed 3-manifold is a connected sum of prime 3-manifolds, uniquely up to order and homeomorphism. Theorem 2.11 appears in [Rolf–04, Theorem 1.1] and [BoRW–05, Theorem 3.7].

Theorem 2.12 ([Sun–15])**.** Let M, N be oriented connected closed 3-manifolds; suppose that M is hyperbolic.

Then there exists a finite covering $\widetilde{M} \longrightarrow M$ and a continuous map $\widetilde{M} \longrightarrow N$ of degree 2. In other words, M virtually 2-dominates every oriented connected closed 3-manifold.

For "most" geometrizable 3-manifolds M, the possible degrees of a continuous map from M to itself are in $\{-1, 0, 1\}$. Exceptions include manifolds covered by a torus bundle over the circle, or covered by $\Sigma_g \times S^1$ for $g \geq 2$, or covered by one of S^3, $S^2 \times \mathbb{R}$. Possible degrees of self-mappings of these "exceptions" are determined in [SWWZ–12].

Attention has also been given to domination between 3-manifolds with boundaries; as a sample, we cite only [BBRW] and references there, on domination of knot exteriors.

To conclude this section, we describe some results of Gaifullin [Gaif–08a, Gaif–08b, Gaif–13] for manifolds of dimensions not necessartily 3. Claim (1) below involves the Tomei manifolds, defined as follows; see [Tome–84], as well as [Davi–87].

For $n \geq 1$, choose $n+1$ pairwise distinct real numbers, say $\lambda_1, \ldots, \lambda_{n+1}$, and define the Tomei manifold M_0^n as the space of symmetric real matrices $(a_{i,j})_{1 \leq i,j \leq n+1}$ which are tridiagonal, i.e. $a_{i,j} = 0$ whenever $|j - i| \geq 2$, and with eigenvalues $\lambda_1, \ldots, \lambda_{n+1}$. It can be shown that

(i) M_0^n is an orientable connected closed smooth n-manifold, and its diffeomorphism type is independent of the choice of the λ_i 's;
(ii) M_0^n is aspherical;
(iii) $\pi_1(M_0^n)$ is explicitely described in [Davi–87] as a torsion-free finite-index subgroup of a Coxeter group with $2n$ generators;
(iv) M_0^1 is the circle, and M_0^2 is an orientable closed surface of genus 2.

Theorem 2.13 (Gaifullin)**.** (1) For every $n \geq 2$, there exists an oriented connected closed n-manifold, indeed the manifold M_0^n described above, such that every oriented connected closed n-manifold is dominated by some finite cover of M_0^n.

(2) For $n \in \{2, 3, 4\}$, there exists a hyperbolic oriented connected closed n-manifold of the form H^n/Γ such that every oriented connected closed n-manifold is dominated by some finite cover of H^n/Γ.

Observe that Claim (1) is straightforward for $n = 2$, and true (but completely trivial) for $n = 1$. In (2), H^n denotes the hyperbolic space of dimension n and constant curvature -1, and Γ a torsion-free cocompact discrete subgroup of the isometry group of H^n. As far as I know, it is unknown whether (2) extends to dimensions $n \geq 5$.

Gaifullin considers also the more general problem of realising homology classes by manifolds (see Remark 4.2 below), but this goes beyond the scope of the present exposition.

3. Simplicial volume

Proposition 3.3 provides another obstruction to domination of one manifold by another. Simplicial volumes appear in [Grom–82]; see also Chapter 6 of [Thur–80], entitled "Gromov's invariant and the volume of a hyperbolic manifold", and [Loh–11].

Definition 3.1. The **simplicial volume** $\|M\|_1$ of an oriented connected closed n-manifold M is the ℓ^1-semi-norm of the fundamental class $[M]$ in $H_n(M;\mathbb{R})$, i.e. the infimum of $\sum_k |h_k|$ over all singular cycles $\sum_k h_k\sigma_k$ representing the fundamental class $[M] \in H_n(M;\mathbb{R})$.

More generally, for a topological space X and an integer $j \geq 0$, every homology class $h \in H_j(X;\mathbb{R})$ has a ℓ^1**-semi-norm** $\|h\|_1$ defined as the infimum of $\sum_k |h_k|$ over all singular cycles $\sum_k h_k\sigma_k$ representing h.

(Here, each σ_k stands for a singular j-simplex, i.e. for a continuous map from the standard j-simplex $\{(x_0,\dots,x_j) \in \mathbb{R}^{j+1} \mid x_0 \geq 0,\dots,x_j \geq 0, \sum_{i=0}^j x_i = 1\}$ to the space X).

Remark 3.2. Dually, a cohomology class $c \in H^j(X;\mathbb{R})$ has a (possibly infinite!) ℓ^∞-semi-norm $\|c\|_\infty$ defined as the infimum of $\|z\|_\infty$ over all singular cocycles z representing c, and $\|z\|_\infty := \sup_\sigma |z(\sigma)|$, where $\sup_\sigma$ stands for the supremum over the set of all singular simplices $\sigma : \Delta^j \longrightarrow X$. (It is natural to ask whether this semi-norm is a norm [Grom–82, Page 38]. The answer is positive for $j \leq 2$ [MaMo–85], but negative in general [Soma–97].)

Let M be as in the previous definition, and $[M]^* \in H^n(M;\mathbb{R})$ its dual fundamental class, as in 1.9. Then $\|[M]^*\|_\infty = (\|M\|_1)^{-1}$; in particular, $[M]^*$ is a bounded class if and only if $\|M\|_1 > 0$ [Grom–82, Corollary on Page 17].

The following basic and elementary proposition appears in [Grom–82, Page 8].

Proposition 3.3. Let M, N be two oriented connected closed manifolds of the same dimension, and $\varphi : M \longrightarrow N$ a continuous map. Then

$$\|M\|_1 \;\geq\; |\deg(\varphi)|\, \|N\|_1.$$

If, moreover, φ is a d-sheeted covering for some $d \geq 1$, then

$$\|M\|_1 \;=\; d\|N\|_1.$$

Proof. The first claim is a particular case of the following straightforward general fact: if $f : X \longrightarrow Y$ is a continuous map between topological spaces and j a non-negative integer, then $\|f_j(c)\|_1 \leq \|c\|_1$ for all $c \in H_j(X;\mathbb{R})$.

Set $n = \dim(N)$. Let $\varepsilon > 0$. There exists a n-cycle $v = \sum_\ell v_\ell\tau_\ell$ representing $[N]$ such that $\|v\|_1 = \sum_\ell |v_\ell| \leq \|N\|_1 + \varepsilon$. Assume that φ is a d-sheeted covering. Since every n-simplex $\tau : \Delta^n \longrightarrow N$ has d lifts $\sigma : \Delta^n \longrightarrow M$, there exists a natural n-cycle

$u = \sum_k u_k \sigma_k \in Z_n(M;\mathbb{R})$ such that $\varphi_n(u) = dv$, and the class of u is $[M]$. We have therefore $\|M\|_1 \le \|u\|_1 = \|dv\|_1 \le |d|\|N\|_1 + |d|\varepsilon$, and the second claim follows. □

Together with 3.5(2), Proposition 3.3 implies:

Corollary 3.4. If $\|M\|_1 = 0$ and N is hyperbolic, then M does not dominate N.

Examples 3.5. Let M, N be oriented connected closed manifolds, and Γ the fundamental group of M.

(1) If there exists a continuous map $M \longrightarrow M$ of degree d with $|d| \ge 2$, then $\|M\|_1 = 0$. For example, $\|S^n\|_1 = 0$ for all $n \ge 1$, see Example 1.10(1).

For $k \ge 2$, the complex projective space $P^n_{\mathbb{C}}$ has a continuous endomorphism of degree $k^n > 1$ given in homogeneous coordinates by

$$(z_0 : z_1 : \cdots : z_n) \longmapsto (z_0^k : z_1^k : \cdots : z_n^k);$$

hence $\|P^n_{\mathbb{C}}\|_1 = 0$ for all $n \ge 1$. Similarly, $\|P^n_{\mathbb{R}}\|_1 = 0$ for all odd $n \ge 3$ (recall that $P^n_{\mathbb{R}}$ is orientable for n odd).

If there exists a nontrivial circle action on M, then $\|M\|_1 = 0$ [Grom–82, Page 41]. In particular, in dimension 3, if M is a Seifert manifold, then $\|M\|_1 = 0$.

(2) If M is hyperbolic, then $\|M\|_1 > 0$. More precisely, if $M = H^n/\Gamma$, where H^n is the hyperbolic space of dimension n (with $n \ge 2$), and Γ an appropriate cocompact discrete subgroup of the group of orientation-preserving isometries of H^n, then

$$\|M\|_1 \,=\, \mathrm{Vol}(M)/\nu_n,$$

where Vol indicates the Riemannian volume and ν_n the maximal volume of an ideal n-simplex in H^n. This is known as the Gromov-Thurston theorem; see e.g. [Thur–80, Theorem 6.2].

For surfaces, let Σ_g denote an orientable connected closed surface of genus g, and thus of Euler characteristic $\chi(\Sigma_g) = 2 - 2g$. Then

$$\|\Sigma_g\|_1 \,=\, 2|\chi(\Sigma_g)| \,=\, 2g-2 \,>\, 0 \quad \text{when } g \ge 2$$

and $\|\Sigma_g\|_1 = 0$ when $g = 0, 1$.

(3) Consider a continuous map $f : \Sigma_g \longrightarrow \Sigma_h$, and let $d = \deg(f)$ denote its degree. We have already observed that, if $h = 0$, then g and d can be arbitrary (Example 1.10(1)).

Assume from now on that $d \ne 0$. By Example 2.8(3), we know that $g \ge h$. If $h = 1$, then $g \ge 1$ and, similarly, d can be arbitrary.

Assume from now on that, moreover, $h \ge 2$. We have $2(g-1) = |\chi(\Sigma_g)| \ge |d|\,|\chi(\Sigma_h)| = |d|2(h-1)$ by Proposition 3.3, i.e.

$$|d| \,\le\, \frac{g-1}{h-1}$$

(a result already in Kneser [Knes–30]). This improves a conclusion in Example 2.8(3). We leave it to the reader to check that, for $g \ge h \ge 2$, every d with $|d| \le \frac{g-1}{h-1}$ is the degree of some continuous map $\Sigma_g \longrightarrow \Sigma_h$.

(4) If M is a compact Riemannian locally symmetric spaces of the non-compact type, then $\|M\|_1 > 0$. This has been conjectured in [Grom–82, Page 11], and proved by the conjunction of [LaSc–06] and [Buch–07].

More generally, Gromov asked if (or conjectured that ?) this holds for every manifold with non-positive curvature and negative Ricci curvature.

(5) For M, N closed connected manifolds of dimension m, n respectively:

$$(\Pi) \qquad \|M\|_1 \|N\|_1 \le \|M \times N\|_1 \le \binom{m+n}{n} \|M\|_1 \|N\|_1 .$$

Note. Let Δ^m be a m-simplex in $\mathbb{R}^m$, with vertices $a_0, a_1, \dots, a_m$, and Δ^n a n-simplex in $\mathbb{R}^n$, with vertices $b_0, b_1, \dots, b_n$. In $\mathbb{R}^{m+n}$, there are $\binom{m+n}{m}$ sequences of the form

$$(a_0, b_0) = (a_{j_0}, b_{k_0}), (a_{j_1}, b_{k_1}), \dots, (a_{j_\ell}, b_{k_\ell}) \dots, (a_{j_{m+n}}, b_{k_{m+n}}) = (a_m, b_n),$$

where (a_{j_ℓ}, b_{k_ℓ}) is followed either by $(a_{j_\ell +1}, b_{k_\ell})$ or by $(a_{j_\ell}, b_{k_\ell +1})$, for $\ell \in \{0, 1, \dots, m+n-1\}$. The $(m+n)$-simplices of this form, $\binom{m+n}{m}$ of them, constitute a triangulation of the product $\Delta^m \times \Delta^n$. This is where the constant $\binom{m+n}{m}$ in (Π) comes from.

(6) Let M be a closed connected manifold such that $\|M\|_1 > 0$. In most of known examples *which are not* hyperbolic manifolds, the exact value of $\|M\|_1$ is not known.

There is one remarkable exception, for manifolds covered by $H^2 \times H^2$. For example, $\|\Sigma_g \times \Sigma_h\|_1 = \frac{3}{2} \|\Sigma_g\|_1 \|\Sigma_h\|_1$ [Buch–08].

(7) If $n \ge 3$ and $M = M_1 \sharp M_2$ is a connected sum of oriented connected closed n-manifolds, then $\|M\|_1 = \|M_1\|_1 + \|M_2\|_1$ [Grom–82, Section 3.5]. Note that this is not true for surfaces ($n = 2$) !

(8) Let M be an oriented connected closed 3-manifold; assume that M is irreducible. Let $H_1, \dots, H_k, S_{k+1}, \dots, S_\ell$ be the pieces of a JSJ-decomposition of M, with $H_1, \dots, H_k$ hyperbolic and $S_{k+1}, \dots, S_\ell$ non-hyperbolic. For $j = 1, \dots, k$, denote by $\mathrm{Vol}(H_j)$ the volume of the hyperbolic structure on the interior of H_j. Then

$$\|M\|_1 \ = \ \frac{1}{\nu_3} \sum_{j=1}^{k} \mathrm{Vol}(H_j),$$

with ν_3 is the maximal volume of ideal 3-simplices, as in (2) above. See [Soma–81], and also [Grom–82, Section 4.2, Corollary on Page 58].

(9) If $\pi_1(M)$ is amenable, then $\|M\|_1 = 0$. In particular, if M is simply connected or with finite fundamental group, then $\|M\|_1 = 0$. See [Grom–82, Section 3.1].

(10) Let M be the total space of a fibre bundle of which both the base space and the fibre are oriented connected closed manifolds of positive dimensions. If the fundamental group of the fibre is amenable, then $\|M\|_1 = 0$. See [Loh–11, Section 5.2].

(11) If a manifold M of dimension at least 2 is rationally essential (Definition 4.4 below) and if its fundamental group is Gromov-hyperbolic, then $\|M\|_1 > 0$. See [Loh–11, Corollary 5.3].

(12) If $n = 4$, and M fibers over an oriented base B with fibers oriented surfaces Σ_g of genus $g \ge 2$, then $\|M\|_1 \ge \|\Sigma_g\|_1 \, \|B\|_1$ [HoKo–01].

(13) Consider the oriented connected closed 3-manifolds $M = S^1 \times \Sigma_g$ and $N = H^3/\Gamma$, where Σ_g denotes an oriented connected closed surface of genus g at least two,

H^3 the hyperbolic space of dimension 3, and Γ an appropriate cocompact discrete subgroup of the group of isometries of H^3.

Since $\|M\|_1 = 0$ and $\|N\|_1 > 0$, the manifold M does not dominate N, by Proposition 3.3 or Example 2.3(8).

Observe however that, for g at least as large as the minimal number of generators of Γ, the obstructions of Proposition 2.6 do not apply.

Indeed, there is a surjection of $\pi_1(\Sigma_g)$ onto the free group F_g of rank g, a fortiori a surjection of $\pi_1(M) = \mathbb{Z} \times \pi_1(\Sigma_g)$ onto F_g, and a surjection of F_g onto $\pi_1(N) = \Gamma$. The composition $\pi_1(M) \longrightarrow \pi_1(N)$ is an epimorphism; compare with Proposition 2.6(1). Also

$$H_1(M;\mathbb{Q}) \simeq (\pi_1(M)/[\pi_1(M), \pi_1(M)]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

surjects onto

$$H_1(N;\mathbb{Q}) \simeq (\pi_1(N)/[\pi_1(N), \pi_1(N)]) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and it follows that the Betti numbers satisfy $\dim(H_j(M;\mathbb{Q})) \geq \dim(H_j(N;\mathbb{Q}))$ for all j; compare with Proposition 2.6(2).

Example 3.6. Here is an example of non-domination which can be proved by other methods, giving rise to other kinds of obstructions.

Let M be a compact Kähler manifold, and N a compact locally symmetric space of the noncompact type which is not locally hermitian symmetric, of the same dimension as M. Then M does not dominate N. This is a particular case of results in [CaTo–89], obtained using Eells and Sampson's theory of harmonic mappings.

For example, if Γ is a torsion-free cocompact lattice in $\mathrm{SL}_n(\mathbb{R})$, with $n > 2$, a compact Kähler manifold does not dominate $\Gamma \backslash \mathrm{SL}_n(\mathbb{R}) / \mathrm{SO}(n)$.

Remark 3.7. There is a "mapping theorem" in bounded cohomology [Grom–82, Section 3.1]: Let X, Y be connected topological spaces which are good enough (say CW-complexes, for simplicity) and $f : X \longrightarrow Y$ a continuous map which induces an isomorphism $f_\pi = \pi_1(X) \longrightarrow \pi_1(Y)$. Then $f^* : H_b^*(Y) \longrightarrow H_b^*(X)$ is an isometric isomorphism (with respect to the norms $\| \cdot \|_\infty$).

It follows that, if M is an oriented connected closed manifold and $c : M \longrightarrow B(\pi_1(M))$ its classifying map, then $\|M\|_1 = \|f_n([M])\|_1$.

Note however that two manifolds with isomorphic fundamental groups *need not* have the same simplicial volume. For example, if M is such that $\|M\|_1 > 0$, then $\|M \times S^2\|_1 = 0$ by Example 3.5(5); of course $\pi_1(M \times S^2) \simeq \pi_1(M)$.

4. Domination by products for manifolds and for groups

Here is a particular case of Question 2.2.

Question 4.1. Let N be an oriented connected closed manifold of dimension n. Can N be dominated by a product ?

In other words, do there exist two oriented connected closed manifolds M_1, M_2, say of dimensions $n_1, n_2 \geq 1$ with $n_1 + n_2 = n$, such that $M_1 \times M_2$ dominates N ?

Proposition 4.6 below shows that, in some cases, Question 4.1 can be translated to fundamental groups.

Remark 4.2. Question 4.1 refers more or less implicitely to other very classical questions.

Steenrod's problem on realization of cycles asks: *for a finite polyhedron K and a homology class $z \in H_j(K;\mathbb{Z})$, does there exist an oriented connected closed manifold M of dimension j and a map $f : M \longrightarrow K$ such that z is the image by f_j of the fundamental class of M ?* see [Eile–49, Problem 25]. The answer is positive when $j \leq 5$; it need not be positive in the general case [Thom–54, théorèmes III.3 and III.9], but that there exists always a multiple kz (where $k \in \mathbb{Z}$ depends on j only) which is of the form $f_j([M])$ [Thom–54, théorème III.4].

There is in [Grom–99, Page 304] a discussion of classes in $H_j(\cdot;\mathbb{Z})$, with j even, which can be represented by products of surfaces.

Reminder 4.3 (Classifying spaces of coverings)**.** Let us review some material on classifying spaces for coverings. To my surprise, the only references I found on this are in the setting of topological groups and principal bundles (see e.g. [Dold–63]), rather than in the simpler setting of groups and coverings. Say here that a topological space is "good enough" if it is path connected, locally path connected, and locally simply connected (so that fundamental group and coverings are simply dealt with).

Let Γ be a group. Recall that there exists a **classifying covering** $\pi_\Gamma : E(\Gamma) \longrightarrow B(\Gamma)$ where $E(\Gamma), B(\Gamma)$ are topological spaces and $B(\Gamma)$ is a pointed space, with the following property. For any space X which is good enough and paracompact and for any covering $p : E \longrightarrow X$ of group Γ, there exists a continuous map (well-defined up to homotopy equivalence) $c : X \longrightarrow B(\Gamma)$ such that the covering p is isomorphic to the pullback by c of the classifying covering π_Γ. The base space $B(\Gamma)$ is the **classifying space for the group** Γ.

It is a basic result that a covering of group Γ over a good enough space is classifying if and only if its total space is contractible. Consequently, another name for "classifying space for Γ" is "Eilenberg-MacLane space $K(\Gamma, 1)$".

Let Δ be another group and $\varphi : \Gamma \longrightarrow \Delta$ a homomorphism. Given a covering $p : E \longrightarrow X$ of group Γ, there is an induced covering $(\varphi)_* p : F \longrightarrow X$ of group Δ, in which the total space F is the quotient $E \times_\Gamma \Delta$ of $E \times \Delta$ by the equivalence relation generated by $(\xi, \delta) \sim (\xi\gamma, \varphi(\gamma)\delta)$, for all $\xi \in E, \delta \in \Delta, \gamma \in \Gamma$.

The assignment to a group of its classifying space is functorial in the following sense: to every homomorphism $\varphi : \Gamma \longrightarrow \Delta$, corresponds a continuous map $B(\varphi) : B(\Gamma) \longrightarrow B(\Delta)$ mapping base point to base point, such that, if $p : E \longrightarrow X$ is a covering of group Γ, pulled back as above from π_Γ by some continuous map $c : X \longrightarrow B(\Gamma)$, then $(\varphi)_* p$ is isomorphic to the covering of group Δ pulled back from π_Δ by the composition $B(\varphi) \circ c$.

One way to prove these claims about the functor $B(\cdot)$ is to invoke Milnor's join construction of $B(\Gamma)$ [Miln–56].

For example, let φ be the reduction modulo 2 from $\Gamma = \mathbb{Z}$ to $\Delta = \mathbb{Z}/2\mathbb{Z}$. The standard model for the classifying space of $\mathbb{Z}$ is the circle $\mathbb{R}/\mathbb{Z}$ (the total space of the classifying covering is $\mathbb{R}$), and that for $\mathbb{Z}/2\mathbb{Z}$ is the infinite real projective space $P^\infty_\mathbb{R} = \lim_n P^n_\mathbb{R}$ (the corresponding total space is an inductive limit $\lim_n S^n$ of spheres). Then $B(\varphi) : \mathbb{R}/\mathbb{Z} \longrightarrow P^\infty_\mathbb{R}$ maps the circle to a loop which represents the non-trivial element of the fundamental group of $P^\infty_\mathbb{R}$.

Let $\Gamma = \Gamma_1 \times \Gamma_2$ be a group given as a direct product of two subgroups. The two projections $\varphi_j : \Gamma \longrightarrow \Gamma_j$ provide a map $B(\Gamma) \longrightarrow B(\Gamma_1) \times B(\Gamma_2)$ *which is an isomorphism*. Indeed, since homotopy groups of a product of spaces are products of homotopy groups, we have $\pi_1(B(\Gamma_1) \times B(\Gamma_2)) = \Gamma_1 \times \Gamma_2 = \Gamma$ and $\pi_j(\text{idem}) = \{0\}$ for $j \geq 2$; hence $B(\Gamma_1) \times B(\Gamma_2)$ is a classifying space for Γ, as claimed.

Consider in particular two connected closed manifolds M, N and a continuous map $f : M \longrightarrow N$. Let $\widetilde{M}$ denote the universal covering of M, which is a covering of group $\pi_1(M)$, pullback of the classifying covering for this group by some continuous map $c_M : M \longrightarrow B(\pi_1(M))$. Similarly, the universal covering $\widetilde{N}$ is pulled back of the classifying covering for $\pi_1(N)$ by some continuous map $c_N : N \longrightarrow B(\pi_1(N))$.

Then the maps $c_N \circ f$ and $B(f_\pi) \circ c_M$ are homotopic.

The following definition is a variation on a notion introduced in [Grom–82, Page 95] and [Grom–83, § 0]. For a group Γ, we write $H_*(\Gamma; \mathbb{Q})$ for $H_*(B(\Gamma); \mathbb{Q})$.

Definition 4.4. An orientable connected closed manifold M of positive dimension n is **rationally essential** if, with the notation above, the image of a fundamental class $[M]$ by the map $(c_M)_n : H_n(M; \mathbb{Q}) \longrightarrow H_n(\pi_1(M); \mathbb{Q})$ is not zero.

Examples 4.5. Let M be an orientable connected closed manifold.

(1) A connected manifold is **aspherical** if its universal covering is contractible; more on these manifolds in [Luck–12]. If M is aspherical, then M is rationally essential, and indeed M itself is a good model for $B(\pi_1(M))$. There are many aspherical examples of orientable connected closed manifolds. The most standard examples include:

(1.i) all surfaces, except the sphere;

(1.ii) all 3-manifolds except those of the three following classes (as a consequence of the sphere theorem of Papakyriakopoulos): manifolds finitely covered by the 3-sphere, non-trivial connected sums (they have non-trivial π_2), and $S^1 \times S^2$;

(1.iii) all manifolds admitting a Riemannian metric with nonpositive sectional curvature (a consequence of the Cartan-Hadamard theorem);

(1.iv) all manifolds of the form $\Gamma \backslash G/K$, with G a connected Lie group, K a maximal compact subgroup of G, and Γ a torsion-free cocompact lattice in G (a consequence of the Cartan-Malcev-Iwasawa theorem); when G is linear semi-simple without compact factors, the corresponding double coset space is an example of (1.iii).

(2) If the simplicial volume $\|M\|_1$ is positive, then M is rationally essential [Grom–82, Section 3.1].

(3) If there exists a rationally essential manifold N and a continuous map $M \longrightarrow N$ of non-zero degree, then M is rationally essential. For example, every connected sum $M_1 \sharp M_2$ with M_2 rationally essential is rationally essential.

(4) If $\pi_1(M)$ is finite, then M is not rationally essential, since $H_j(\pi_1(M); \mathbb{Q}) = \{0\}$ for all $j \geq 1$.

Proposition 4.6 (Kotschick-Löh [KoLo–09])**.** Let M, N be two oriented connected closed manifolds and $f : M \longrightarrow N$ a continuous map; set $\Gamma = \pi_1(M)$. Assume that $M = M_1 \times M_2$ is a non-trivial product. For $j = 1, 2$,

- denote by $n_j \geq 1$ the dimension of M_j;
- choose a base point $x_j \in M_j$;
- set $\Gamma_j = \pi_1(M_j, x_j)$, so that $\Gamma = \Gamma_1 \times \Gamma_2$;
- denote by $f_j : M_j \longrightarrow N$ ($j = 1, 2$) the restriction of f to $M_1 \times \{x_2\}$, $\{x_1\} \times M_2$ respectively;
- denote by Δ_j the image $(f_j)_\pi(\Gamma_j)$.

Observe that Δ_1 and Δ_2 are commuting subgroups of $\pi_1(N)$.

· Denote by ψ the multiplication homomorphism

$$\Delta_1 \times \Delta_2 \longrightarrow \pi_1(N), \ \ (\delta_1, \delta_2) \longmapsto \delta_1\delta_2,$$

so that the image of ψ coincides with the image of f_π.

Suppose moreover that N is rationally essential.

If the degree of f is not zero, then both Δ_1 and Δ_2 are *infinite* subgroups of $\pi_1(N)$.

Proof. Recall that f_π denotes the homomorphism $\pi_1(M) \longrightarrow \pi_1(N)$ induced by f. We have a diagram

$$\begin{tikzcd}
M_1 \times M_2 = M \arrow[r, "c_{M_1} \times c_{M_2} = c_M"] \arrow[dd, "f"] & B(\Gamma_1) \times B(\Gamma_2) \arrow[r, "="] \arrow[d, "B((f_1)_\pi) \times B((f_2)_\pi)"] & B(\Gamma) \arrow[dd, "B(f_\pi)"] \\
 & B(\Delta_1) \times B(\Delta_2) \arrow[d, "B(\psi)"] & \\
N \arrow[r, "c_N"] & B(\pi_1(N)) \arrow[r, "="] & B(\pi_1(N))
\end{tikzcd}$$

commuting up to homotopy. (We have abusively denoted by "=" several canonical isomorphisms.)

There is a corresponding commutative diagram for the homologies $H_n(\cdot; \mathbb{Q})$, in which there are three ways to go from the top left corner to the bottom right corner.

(i) Going through the left-hand vertical arrow, the fundamental class $[M]$ in $H_n(M; \mathbb{Q})$ is mapped to $\deg(f)(c_N)_n([M])$ in $H_n(\pi_1(N); \mathbb{Q})$.

(ii) Going through the central vertical arrow, $[M]$ goes through a class $h_1 \otimes h_2 \in H_{n_1}(\Delta_1; \mathbb{Q}) \otimes H_{n_2}(\Delta_2; \mathbb{Q}) \subset H_n(\Delta_1 \times \Delta_2; \mathbb{Q})$.

(iii) Going through the right-hand vertical arrow, $[M]$ goes to

$$(B(f_\pi))_n \circ (c_M)_n ([M]) \ \in \ (B(f_\pi))_n \left(H_n(\Gamma; \mathbb{Q})\right) \ \subset \ H_n(\pi_1(N); \mathbb{Q}).$$

These three images of $[M]$ in $H_n(\pi_1(N); \mathbb{Q})$ coincide.

If $\deg(f) \neq 0$ and if N is rationally essential, this common image is not zero. Hence $h_1 \otimes h_2 \neq 0$; in particular $H_{n_1}(\Delta_1; \mathbb{Q}) \neq \{0\}$ and $H_{n_2}(\Delta_2; \mathbb{Q}) \neq \{0\}$. It follows that Δ_1 and Δ_2 are infinite groups. □

This suggests the following definition, again from [KoLo–09]:

Definition 4.7. A group Δ is **presentable by a product**, or shortly below **pbp**, if there exist two groups Γ_1, Γ_2 and a homomorphism $\varphi : \Gamma_1 \times \Gamma_2 \longrightarrow \Delta$ with the two following properties:

(i) both $\varphi(\Gamma_1)$ and $\varphi(\Gamma_2)$ are infinite subgroups of Δ,

(ii) these subgroups generate in Δ a subgroup of finite index.

Examples 4.8. (1) A group Γ with infinite centre is pbp, because the multiplication $\Gamma \times Z(\Gamma) \longrightarrow \Gamma$ has Properties (i) and (ii) above.

(2) Let Δ be a group and Δ' a subgroup of finite index. Then Δ is pbp if and only if Δ' is pbp.

(3) Let $\Delta = \Delta_1 * \Delta_2$ be a free product of two non-trivial groups. Then Δ is pbp if and only if both Δ_1 and Δ_2 are of order 2 (and therefore Δ is an infinite dihedral group). See [KoLo–13, Corollary 9.2].

(4) An infinite simple group is not pbp.

(5) A non-elementary Gromov hyperbolic group is not pbp [KoLo–09, Theorem 1.5].

(6) For a closed 3-manifold M with infinite fundamental group, the following properties are equivalent [KoNe–13, Theorem 8]:
(i) $\pi_1(M)$ is pbp,
(ii) $\pi_1(M)$ has a finite index subgroup with infinite centre,
(iii) M is a Seifert manifold.

For other examples, see [KoLo–09, KoLo–13, HaKo], as well as Section 5 below.

With the terminology of Definition 4.7, Proposition 4.6 can be reformulated as follows [KoLo–09, Theorem 1.4]:

Theorem 4.9 (Kotschick-Löh)**.** Let N be an oriented connected closed manifold. Assume that N is rationally essential and that $\pi_1(N)$ is not pbp. Then N is not dominated by any non-trivial product of closed oriented connected manifolds.

In the converse direction, let us quote the following result [Neof, Theorem A]:

Theorem 4.10 (Neofytidis)**.** Let N be an oriented connected closed manifold. Assume that N is aspherical and that there is a subgroup of finite index in $\pi_1(N)$ that is a direct product of two infinite groups. Then N is dominated by a non-trivial direct product of oriented connected closed manifolds.

In contrast, there exist oriented connected closed 4-manifolds M such that $\pi_1(M)$ is the direct product of two infinite groups, and yet M is not dominated by any non-trivial product [KoLo–09, Exampe 6.3]. Of course, these examples cannot be aspherical.

5. Some examples, including Coxeter groups

In this section, k denotes a field of characteristic 0. For more details, see [HaKo].
Definition 4.7 has analogues in other categories, e.g.:

Definition 5.1. A k-Lie algebra $\mathfrak{g}$ is **presentable by a product** if there exist two commuting subalgebras $\mathfrak{g}_1, \mathfrak{g}_2$ of $\mathfrak{g}$ of positive dimensions such that the homomorphism $\mathfrak{g}_1 \times \mathfrak{g}_2 \longrightarrow \mathfrak{g}$, $(X_1, X_2) \longmapsto X_1 + X_2$, is surjective.
Note that, if $\mathfrak{g}_1, \mathfrak{g}_2$ are as above, then they are non-zero ideals in $\mathfrak{g}$, and each one is contained in the centralizer in $\mathfrak{g}$ of the other.

Definition 5.2. A connected linear algebraic k-group $\mathbf{G}$ is **presentable by a product** if there exist two commuting connected closed k-subgroups $\mathbf{G}_1, \mathbf{G}_2$ of $\mathbf{G}$ of positive dimensions such that the multiplication homomorphism $\mu : \mathbf{G}_1 \times \mathbf{G}_2 \longrightarrow \mathbf{G}$, $(g_1, g_2) \longmapsto g_1 g_2$, is surjective.
Note that, if $\mathbf{G}_1, \mathbf{G}_2$ are as above, then they are non-trivial normal subgroups of $\mathbf{G}$, and each one is contained in the centralizer of the other.

Proposition 5.3. Let $\mathbf{G}$ be a connected linear algebraic k-group, $\mathfrak{g}$ its Lie algebra, and Γ a Zariski dense subgroup of the group $G = \mathbf{G}(k)$ of rational points of $\mathbf{G}$.
(1) If $\mathfrak{g}$ is not presentable by a product, then $\mathbf{G}$ is not presentable by a product.

(2) If $\mathbf{G}$ is not presentable by a product, then Γ is not presentable by a product. [*The proof is routine; we refer to* [HaKo].]

Examples 5.4. (1) Let $\mathrm{Af}_1(\mathbb{R}) = \begin{pmatrix} \mathbf{R}^\times & \mathbf{R} \\ 0 & 1 \end{pmatrix}$ be the real affine group, or more precisely the group of real points of the appropriate algebraic group. Observe that $\mathrm{Af}_1(\mathbb{R})$ is centreless, of dimension 2. As an algebraic group, it is connected, even if, as a real Lie group, its connected component $\begin{pmatrix} \mathbf{R}^\times_+ & \mathbf{R} \\ 0 & 1 \end{pmatrix}$ is of index two.

Its Lie algebra is the non-abelian soluble Lie algebra $\mathfrak{af}_1(\mathbb{R})$ of dimension 2, with basis $\{e, g\}$ such that $[g, e] = e$.

For every $n \in \mathbb{Z}$ with $|n| \geq 2$, the soluble Baumslag-Solitar group $\mathrm{BS}(1, n)$, of presentation $\langle a, t \mid tat^{-1} = a^n \rangle$, is isomorphic to a Zariski dense subgroup of $\mathrm{Af}_1(\mathbb{R})$:

$$\mathrm{BS}(1, n) \approx \begin{pmatrix} \mathbf{n}^{\mathbf{Z}} & \mathbf{Z}[1/n] \\ 0 & 1 \end{pmatrix} \subset \begin{pmatrix} \mathbf{R}^\times & \mathbf{R} \\ 0 & 1 \end{pmatrix} = \mathrm{Af}_1(\mathbb{R}).$$

Since $\mathfrak{af}_1(\mathbb{R})$ has a unique non-trivial ideal, $\mathbb{R}e$ of dimension one, $\mathfrak{af}_1(\mathbb{R})$ is not presentable by a product. It follows from Proposition 5.3 that the group $\mathrm{BS}(1, n)$ is not presentable by a product.

It is a fact that all Baumslag-Solitar groups $\mathrm{BS}(m, n)$ with $|m| \neq |n|$ are not presentable by products [HaKo].

(2) For $n \geq 2$, the group $\mathrm{SL}_n(\mathbb{Z})$ is not presentable by a product. More generally, every Zariski dense subgroup of $\mathrm{SL}_n(\mathbb{R})$ is not presentable by a product.

(3) Consider a semi-direct product $\mathfrak{h} = V^r \rtimes \mathfrak{g}$, where V is a finite dimensional k-vector space, $\mathfrak{g}$ a simple Lie subalgebra of $\mathfrak{gl}(V)$ acting irreducibly on V, and r a positive integer; the space $V^r = V \oplus \cdots \oplus V$ (with r factors) is considered as an abelian Lie algebra on which $\mathfrak{g}$ acts by the restriction of the natural diagonal action of $\mathfrak{gl}(V)$ on V^r. Then $\mathfrak{h}$ is not presentable by a product (Hint: every non-zero ideal of $V^r \rtimes \mathfrak{g}$ is either the full Lie algebra, with centre $\{0\}$, or a $\mathfrak{g}$-invariant subspace of V^r, with centralizer V^r.)

Consider now a Coxeter system (W, S), with S finite. Let B_S denote the Tits form on the vector space $\mathbb{R}^S$, and $\mathrm{Of}(B_S)$ the group of invertible linear transformations $g \in \mathrm{GL}(\mathbb{R}^S)$ such that $B_S(gv, gw) = B_S(v, w)$ for all $v, w \in \mathbb{R}^S$ and $gv = v$ for all $v \in \ker(B_S)$.

Suppose that (W, S) is irreducible, and neither finite nor affine. On the one hand, it is easy to check that its Lie algebra $\mathfrak{of}(B_S)$ is of the form of $\mathfrak{h}$ in Example 5.4(3), so that it is not presentable by a product. On the other hand, the geometric representation $\sigma_S : W \longrightarrow \mathrm{Of}(B_S)$ is Zariski-dense [BeHa–04]. It follows that W is not presentable by a product. To summarize:

Example 5.5. Let (W, S) be a Coxeter system, with S finite, which is irreducible, infinite, and non-affine. Then W is not presentable by a product.

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