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Cubical simplicial volume*

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ABSTRACT. Cubical simplicial volume is a variation on simplicial volume, based on cubes instead of simplices. Both invariants are homotopy invariants of oriented closed connected manifolds. In this note, we prove that cubical simplicial volume of oriented closed connected surfaces is proportional to ordinary simplicial volume. More precisely, the cubical simplicial volume of an oriented closed connected surface of genus $g > 0$ is equal to $2 \cdot g - 2$.

55N10, 57N65

1. INTRODUCTION

Simplicial volume is a homotopy invariant of manifolds measuring the minimal complexity of singular fundamental cycles with $\mathbb{R}$-coefficients [3, 9, 6]. Similarly, cubical singular chains lead to cubical simplicial volume (see Section 2 for the definitions).

In this note, we prove that cubical simplicial volume of oriented closed connected surfaces is proportional to ordinary simplicial volume:

Theorem 1.1. Let S be an oriented closed connected surface of genus $g > 0$. Then

$$\|S\|^{\square} = 2 \cdot g - 2 = \frac{1}{2} \cdot \|S\|^{\triangle}.$$

As in the case of simplicial volume [3, 1] the estimate $\|S\|^{\square} \leq 2 \cdot g - 2$ can be obtained from a corresponding estimate for integral cubical simplicial volume and passage to finite coverings (Section 4).

Theorem 1.2. Let S be an oriented closed connected surface of genus $g > 0$. Then

$$\|S\|^{\square}_{\mathbb{Z}} \leq 2 \cdot g - 1 = \frac{1}{2} \cdot \|S\|^{\triangle}_{\mathbb{Z}}.$$

Conversely, subdividing singular squares into two singular simplices shows that $\|S\|^{\square} \geq 1/2 \cdot \|S\|^{\triangle} = 1/2 \cdot (4 \cdot g - 4)$ (Section 3). Putting both estimates together proves Theorem 1.1.

Using the classification of 3-manifolds, one can show that also in dimension 3 simplicial volume and cubical simplicial volme are proportional [7]. In higher dimensions, the question whether cubical and ordinary simplicial volume are proportional is an open problem [4, 5.40].

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2. CUBICAL SIMPLICIAL VOLUME

We briefly review the definition of simplicial volume and cubical simplicial volume and introduce some notation.

2.1. **Simplicial volume.** We denote the singular chain complex and the singular homology groups by $C_*^{\triangle}$ and $H_*^{\triangle}$, respectively. For $R \in \{\mathbb{Z}, \mathbb{R}\}$ we write $|\cdot|_{1,R}^{\triangle}$ for the ℓ^1-"norm" on $C_*^{\triangle}(\,\cdot\,;R)$ associated with the R-basis $S_*(\,\cdot\,)$ of singular simplices; notice that in the case $R = \mathbb{Z}$ we only have homogeneity with respect to positive integers. Moreover, if X is a topological space, we write

$$\begin{aligned} \|\cdot\|_{1,R}^{\triangle} \colon H_*^{\triangle}(X;R) &\longrightarrow R_{\geq 0} \\ \alpha &\longmapsto \inf\bigl\{|c|_{1,R}^{\triangle} \bigm| c \in C_*^{\triangle}(X;R) \text{ is a cycle representing } \alpha\bigr\} \end{aligned}$$

for the induced semi-"norm"; in the case $R = \mathbb{Z}$, this semi-norm in general will not be homogeneous. If M is an oriented closed connected manifold and if R is a ring with unit, we denote the corresponding fundamental class by $[M]_R^{\triangle} \in H_*^{\triangle}(M;R)$.

Definition 2.1. Let M be an oriented closed connected manifold. Then the *simplicial volume* and *integral simplicial volume* of M are defined by

$$\begin{aligned} \|M\|^{\triangle} &:= \bigl\|[M]_{\mathbb{R}}^{\triangle}\bigr\|_{1,\mathbb{R}}^{\triangle} \\ \|M\|_{\mathbb{Z}}^{\triangle} &:= \bigl\|[M]_{\mathbb{Z}}^{\triangle}\bigr\|_{1,\mathbb{Z}}^{\triangle}. \end{aligned}$$

Example 2.2. The existence of self-maps of non-zero degree implies that $\|S^n\|^{\triangle} = 0$ for all $n \in \mathbb{N}_{>0}$ and that $\|S^1 \times S^1\|^{\triangle} = 0$ [3][6, Corollary 2.2].

Moreover, $\|S^2\|_{\mathbb{Z}}^{\triangle} = 2$: Because Δ^2 has an odd number of faces, no singular 2-cycle can consist of a single singular simplex; on the other hand, one can easily construct fundamental cycles of S^2 that consist of two singular simplices (with opposite signs).

It is easy to see that simplicial volume and integral simplicial volume are homotopy invariants of oriented closed connected manifolds. On the other hand, in the presence of negative curvature, simplicial volume is related to the Riemannian volume, which leads to geometric applications of simplicial volume [3, 11, 6].

2.2. **Cubical simplicial volume.** We quickly recall the definition of cubical singular homology [8, 2]: Replacing standard simplices with standard cubes leads to cubical singular homology: For $n \in \mathbb{N}$ let $\square^n := [0,1]^n$ denote the *standard n-cube*. If X is a topological space, then continuous maps of type $\square^n \longrightarrow X$ are called *singular n-cubes of X*. The geometric/combinatorial boundary of $\square^n$ consists of $2 \cdot n$ cubical faces. If R is a ring with unit, then a suitable alternating sum of these faces allows to define a chain complex $Q_*(X;R)$ of cubical singular chains with R-coefficients (Figure 1). A singular cube is called *degenerate* if it factors over one of the coordinate projections. Dividing out the subcomplex $D_*(X;R)$ generated by degenerate singular cubes leads to the *cubical chain complex*

$$C_*^{\square}(X;R) := Q_*(X;R)/D_*(X;R)$$

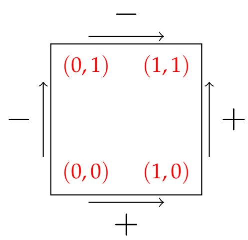

FIGURE 1. The boundary of the 2-cube consists of four 1-cubes (with the indicated parametrisations and signs)

and hence to *cubical singular homology* $H^\square_*(X;R)$ (which admits a natural extension to a functor). Notice that dividing out degenerate singular cubes is necessary in order for cubical singular homology of a point to be concentrated in degree 0.

For $R \in \{\mathbb{Z}, \mathbb{R}\}$ we write $|\cdot|^\square_{1,R}$ for the ℓ^1-"norm" on $Q_*(\,\cdot\,;R)$ associated with the R-basis of cubical singular simplices. If X is a topological space, we define the *cubical ℓ^1-semi-norm*

$$\begin{aligned} \|\cdot\|^\square_{1,R} \colon H^\square_*(X;R) &\longrightarrow R_{\geq 0} \\ \alpha &\longmapsto \inf\bigl\{|c|^\square_{1,R} \bigm| \begin{array}{l} c \in Q_*(X;R) \text{ is a cycle} \\ \text{with } [p_{X,R}(c)] = \alpha \in H^\square_*(X;R)\bigr\}, \end{array} \end{aligned}$$

where $p_{X,R} \colon Q_*(X;R) \longrightarrow C^\square_*(X;R)$ is the canonical projection. I.e., we only look at strict cubical cycles that represent the given class.

On the other hand, we can also consider the ℓ^1-"norm" $|\cdot|^\blacksquare_{1,R}$ on $C^\square_*(X;R)$ associated with the R-basis of *non-degenerate* cubical singular simplices. If X is a topological space, we then write

$$\begin{aligned} \|\cdot\|^\blacksquare_{1,R} \colon H^\square_*(X;R) &\longrightarrow R_{\geq 0} \\ \alpha &\longmapsto \inf\bigl\{|c|^\blacksquare_{1,R} \bigm| c \in C^\square_*(X;R) \text{ is a cycle representing } \alpha\bigr\} \end{aligned}$$

for the induced semi-"norm".

Proposition 2.3 (strict vs. degenerate cubical ℓ^1-semi-norm)**.** Let X be a topological space and let $\alpha \in H^\square_*(X;\mathbb{R})$. Then

$$\|\alpha\|^\blacksquare_{1,\mathbb{R}} = \|\alpha\|^\square_{1,\mathbb{R}}.$$

Proof. By definition, $\|\cdot\|^\blacksquare_{1,\mathbb{R}} \leq \|\cdot\|^\square_{1,\mathbb{R}}$. For the converse estimate we look at the following symmetrisation: For $n \in \mathbb{N}$ let $\Sigma^\square_n$ be the isometry group of the Euclidean n-cube $\square^n$ and let

$$\begin{aligned} \Sigma_{X,n} \colon Q_n(X;\mathbb{R}) &\longrightarrow Q_n(X;\mathbb{R}) \\ \operatorname{map}(\square^n, X) \ni c &\longmapsto \frac{1}{|\Sigma^\square_n|} \cdot \sum_{\pi \in \Sigma^\square_n} (-1)^{\operatorname{sgn}\pi} \cdot c \circ \pi; \end{aligned}$$

here, sgn encodes the orientation behaviour, i.e.,

$$\begin{aligned} \mathrm{sgn}\colon \Sigma_n^\square &\longrightarrow \{0,1\} \\ \pi &\longmapsto \begin{cases} 0 & \text{if } \pi \text{ is orientation preserving} \\ 1 & \text{if } \pi \text{ is orientation reversing.} \end{cases} \end{aligned}$$

A straightforward calculation shows that $\Sigma_X\colon Q_*(X;\mathbb{R}) \longrightarrow Q_*(X;\mathbb{R})$ is a chain map. By construction, $\|\Sigma_X\| \leq 1$ with respect to $|\cdot|_{1,\mathbb{R}}^\square$. Furthermore, a reflection argument shows that Σ_X maps degenerate singular cubes to 0. Hence, Σ_X factors over $C_*^\square(X;\mathbb{R})$ and composition with the projection $p_X\colon Q_*(X;\mathbb{R}) \longrightarrow C^\square(X;\mathbb{R})$ yields a chain map

$$\overline{\Sigma}_X\colon C_*^\square(X;\mathbb{R}) \longrightarrow Q_*(X;\mathbb{R}) \longrightarrow C_*^\square(X;\mathbb{R}).$$

Clearly, $\|\overline{\Sigma}_X\| \leq 1$ with respect to $|\cdot|_{1,\mathbb{R}}^\blacksquare$. Moreover, it is not hard to see that $H_*(\overline{\Sigma}_X) = \mathrm{id}_{H_*^\square(X;\mathbb{R})}$. Therefore, $\|\cdot\|_{1,\mathbb{R}}^\square \leq \|\cdot\|_{1,\mathbb{R}}^\blacksquare$. □

It is well known that there is a canonical natural (both in spaces and coefficients) isomorphism $H_*^\square \longrightarrow H_*^\triangle$ [2, Theorem V]. However, in general this isomorphism is not isometric with respect to the corresponding ℓ^1-semi-norms.

If M is an oriented closed connected manifold and if R is a ring with unit, we denote the corresponding fundamental class in cubical singular homology by $[M]_R^\square \in H_*^\square(M;R)$.

Definition 2.4. Let M be an oriented closed connected manifold. Then the *cubical simplicial volume* and *integral cubical simplicial volume* of M are defined by

$$\begin{aligned} \|M\|^\square &:= \big\|[M]_\mathbb{R}^\square\big\|_{1,\mathbb{R}}^\square \\ \|M\|_\mathbb{Z}^\square &:= \big\|[M]_\mathbb{Z}^\square\big\|_{1,\mathbb{Z}}^\square. \end{aligned}$$

In view of Proposition 2.3 we have $\|M\|^\square = \|[M]_\mathbb{R}^\square\|_{1,\mathbb{R}}^\blacksquare$ for all oriented closed connected manifolds M. It is not clear whether the same also holds with integral coefficients.

Example 2.5. Again, self-maps show that $\|S^n\|^\square = 0$ for all $n \in \mathbb{N}_{>0}$ and that $\|S^1 \times S^1\|^\square = 0$.

Wrapping a square around S^2 and mapping the whole boundary of $\square^2$ to a single point shows that $\|S^2\|_\mathbb{Z}^\square = 1$.

3. ESTIMATING CUBICAL SIMPLICIAL VOLUME OF SURFACES FROM BELOW

We will now provide the estimate for cubical simplicial volume of surfaces from below in terms of ordinary simplicial volume. To this end, we subdivide singular squares into two singular simplices:

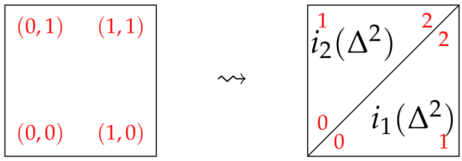

FIGURE 2. Subdividing a square into two triangles

Lemma 3.1 (subdivision of squares)**.**

(1) Let X be a topological space and let R be a ring with unit. Then the natural map

$$\begin{aligned} \varphi_{X,R} \colon Q_2(X;R) &\longrightarrow C_2^{\triangle}(X;R) \\ \operatorname{map}(\square^2, X) \ni \kappa &\longmapsto \kappa \circ i_1 - \kappa \circ i_2 \end{aligned}$$

induces a well-defined natural map

$$\begin{aligned} \Phi_{X,R} \colon H_2\bigl(Q_*(X;R)\bigr) &\longrightarrow H_2^{\triangle}(X;R) \\ [c] &\longmapsto \bigl[\varphi_X(c)\bigr]. \end{aligned}$$

Here, $i_1 \cdot \Delta^2 \longrightarrow \square^2$ and $i_2 \cdot \Delta^2 \longrightarrow \square^2$ denote the affine inclusion of the "lower" and "upper" triangle into the square as indicated in Figure 2.

(2) In particular, we obtain a well-defined natural map

$$\overline{\Phi}_{X,\mathbb{R}} := \Phi_{X,\mathbb{R}} \circ H_2(\Sigma_X) \colon H_2^{\square}(X;\mathbb{R}) \longrightarrow H_2^{\triangle}(X;\mathbb{R})$$

(3) If S is an oriented closed connected surface, then $\overline{\Phi}_{S,\mathbb{R}}$ maps fundamental classes to fundamental classes, i.e., $\overline{\Phi}_{S,\mathbb{R}}([S]^{\square}_{\mathbb{R}}) = \pm[S]^{\triangle}_{\mathbb{R}}$.

Proof. A straightforward calculation shows that $\varphi_{X,R}$ maps strict cubical cycles to ordinary singular cycles. Moreover, a suitable subdivision and orientation of cubes into two prisms and hence six tetrahedra witnesses that $\varphi_{X,R}$ maps boundaries to boundaries. This shows the first part.

The second part follows directly fromt the first part.

The third part follows by considering the local case of $(\mathbb{R}^2, \mathbb{R}^2 \setminus \{0\})$, for which the corresponding statement is easily seen to be true. □

Proposition 3.2 (estimate from below)**.** Let S be an oriented closed connected surface of genus $g > 0$. Then

$$\|S\|^{\square} \geq \frac{1}{2} \cdot \|S\|^{\triangle} = 2 \cdot g - 2.$$

Proof. Clearly, we have $\|\varphi_{S,\mathbb{R}}(c)\|^{\triangle}_{1,\mathbb{R}} \leq 2 \cdot \|c\|^{\square}_{1,\mathbb{R}}$ for all chains c in $Q_2(S;\mathbb{R})$. Therefore, Proposition 2.3, Lemma 3.1, and $\|\Sigma_S\| \leq 1$ imply

$$\|S\|^{\square} = \bigl\|[S]^{\square}_{\mathbb{R}}\bigr\|^{\blacksquare}_{1,\mathbb{R}} \geq \frac{1}{2} \cdot \|S\|^{\triangle}.$$

Moreover, it is well known that $\|S\|^{\triangle} = 4 \cdot g - 4$ holds [3, 1]. □

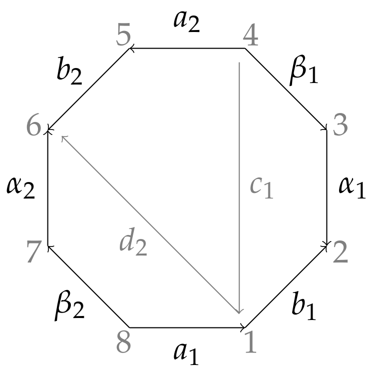

FIGURE 3. A representation of the oriented closed connected surface of genus g, and a cubical fundamental cycle

4. ESTIMATING CUBICAL SIMPLICIAL VOLUME OF SURFACES FROM ABOVE

4.1. **Integral cubical simplicial volume of surfaces.** The core of the proof of Theorem 1.2 is constructing explicit integral cubical fundamental cycles of surfaces:

Proof of Theorem 1.2. By the classification of surfaces, we can view an oriented closed connected surface S of genus $g > 0$ as a quotient of the regular $4 \cdot g$-gon X_g, where the edges $a_1, b_1, \alpha_1, \beta_1, \dots, a_g, b_g, \alpha_g, \beta_g$ are oriented as in Figure 3 and each latin edge is glued to the corresponding greek edge. Moreover, we enumerate the vertices of X_g as indicated in Figure 3 from 1 to $4 \cdot g$. We introduce the following diagonals: For $j \in \{1, \dots, g\}$ let c_j be the diagonal from vertex $4 \cdot j$ to vertex 1; for $j \in \{2, \dots, g\}$ let d_j be the diagonal from vertex 1 to vertex $4 \cdot j - 2$. Notice that $c_g = a_1$.

For a label a of directed edges/diagonals in X_g as above, we define the singular 1-cube $\square_a \colon \square^1 \longrightarrow X_g$ as the linear parametrisation of the directed segment corresponding to a. For labels a, b, c, d of directed edges/diagonals in X_g that form a quadrilateral in X_g oriented as in Figure 4, we define the singular 2-cube

$$\square_{a,b,c,d} \colon \square^2 \longrightarrow X_g$$

as a parametrisation of the convex quadrilateral given by a, b, c, d that induces on the boundary the linear parametrisation on the faces a, b, c, d, i.e., such that

$$\partial^\square(\square_{a,b,c,d}) = \square_a + \square_b - \square_c - \square_d$$

holds.

We then consider the cubical 2-chain

$$\square_{c_1,b_1,\alpha_1,\beta_1} + \sum_{j=2}^{g} (\square_{a_j,b_j,d_j,c_{j-1}} + \square_{c_j,d_j,\alpha_j,\beta_j}) \in Q_2(X_g; \mathbb{Z})$$

and the corresponding induced cubical 2-chain s_g on S. A straightforward calculation shows that s_g indeed is a cycle in $Q_*(S; \mathbb{Z})$ (the singular 1-cubes on the quotient S induced from a latin letter and the corresponding greek letter coincide because of the gluing).

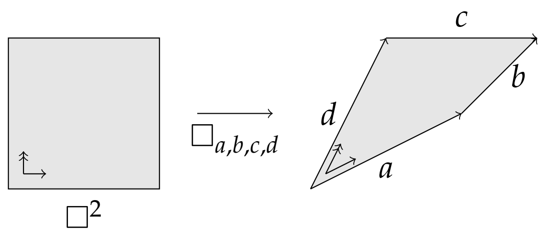

FIGURE 4. A convex quadrilateral with labels and directed edges

Looking at the induced class in $H_2(S, S \setminus \{x\}; \mathbb{Z})$ for some point $x \in S$ that does not lie on any face of a cube in s_g shows that $[p_{S,\mathbb{Z}}(s_g)] = \pm[S]^{\square}_{\mathbb{Z}} \in H^{\square}_2(S; \mathbb{Z})$. Hence,

$$\|S\|^{\square}_{\mathbb{Z}} \leq |s_g|^{\square}_{1,\mathbb{Z}} = 2 \cdot g - 1.$$

It remains to show that $\|S\|^{\triangle}_{\mathbb{Z}} = 4 \cdot g - 2$: By a similar construction as above, it is well known that $\|S\|^{\triangle}_{\mathbb{Z}} \leq 4 \cdot g - 2$ holds [3, 1]. However, no fundamental cycle on S can realise the optimal value $\|S\|^{\triangle} = 4 \cdot g - 4$ because no straight singular simplex on $\mathbb{H}^2$ has maximal volume [5]; hence, $\|S\|^{\triangle}_{\mathbb{Z}} > 4 \cdot g - 4$. Counting the number of faces in a singular chain with $\mathbb{Z}$-coefficients shows (because Δ^2 has an odd number of faces) that $\sum_{j=0}^{k} a_j$ is even for any $\mathbb{Z}$-cycle $\sum_{j=0}^{k} a_j \cdot \sigma_j \in C_2(S; \mathbb{Z})$. Hence, also $\sum_{j=0}^{k} |a_j|$ is even, and so $\|S\|^{\triangle}_{\mathbb{Z}} \geq 4 \cdot g - 2$. $\square$

4.2. **Passage to finite coverings.** We will now combine Theorem 1.2 with multiplicativity of cubical simplicial volume under finite coverings to prove the estimate from above (Proposition 4.2).

Proposition 4.1 (multiplicativity under finite coverings)**.** Let $N \longrightarrow M$ be a finite covering of oriented closed connected manifolds, and let $d \in \mathbb{N}$ be the number of sheets. Then

$$\|M\|^{\square} = \frac{1}{d} \cdot \|N\|^{\square}.$$

Proof. This can be shown literally in the same way as the corresponding multiplicativity for ordinary simplicial volume [10, Corollary 1.19]: The estimate "$\leq$" follows because d-sheeted covering maps have mapping degree $\pm d$; the converse estimate "$\geq$" follows via transfer of cubical singular cycles. $\square$

Proposition 4.2 (estimate from above)**.** Let S be an oriented closed connected surface of genus $g > 0$. Then

$$\|S\|^{\square} \leq 2 \cdot g - 2.$$

Proof. For every $d \in \mathbb{N}_{>0}$ there is a d-sheeted covering $S(d) \longrightarrow S$ of S by an oriented closed connected surface $S(d)$; such coverings can be constructed geometrically or can be obtained from subgroups of $\pi_1(S)$ of index d. Because the Euler characteristic is multiplicative under finite coverings, we obtain that $S(d)$ has genus

$$g(d) := d \cdot (g-1) + 1.$$

From Proposition 4.1 and Theorem 1.2 we therefore obtain

$$\begin{aligned}\|S\|^{\square} &= \frac{1}{d} \cdot \|S(d)\|^{\square} \leq \frac{1}{d} \cdot \|S(d)\|^{\square}_{\mathbb{Z}} \leq \frac{1}{d} \cdot \bigl(2 \cdot g(d) - 1\bigr) \\ &= 2 \cdot g - 2 + \frac{1}{d}.\end{aligned}$$

Taking the infimum over all $d \in \mathbb{N}_{>0}$ gives the desired estimate. $\square$

REFERENCES

[1] R. Benedetti, C. Petronio. *Lectures on Hyperbolic Geometry*, Universitext, Springer, 1992.

[2] S. Eilenberg, S. MacLane. Acyclic models, *Amer. J. Math.*, 75, pp. 189–199, 1953.

[3] M. Gromov. Volume and bounded cohomology, *Inst. Hautes Études Sci. Publ. Math.*, 56, pp. 5–99, 1983.

[4] M. Gromov. *Metric Structures for Riemannian and Non-Riemannian Spaces*, with appendices by M. Katz, P. Pansu, and S. Semmes, translated by S. M. Bates. *Progress in Mathematics*, 152, Birkhäuser, 1999.

[5] D. Jungreis. Chains that realize the Gromov invariant of hyperbolic manifolds, *Ergodic Theory Dynam. Systems*, 17(3), pp. 643–648, 1997.

[6] C. Löh. Simplicial Volume, *Bull. Man. Atl.*, pp. 7–18, 2011

[7] C. Löh, C. Pagliantini, S. Waeber. Cubical simplicial volume of 3-manifolds, preprint, 2015. arXiv: 1508.03017 [math.GT]

[8] W.S. Massey. *A basic course in algebraic topology*, vol. 127 of *Graduate Texts in Mathematics*, Springer, 1991.

[9] H. J. Munkholm. Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem (following Thurston). In *Topology Symposium, Siegen 1979*, *Lecture Notes in Mathematics*, 788, pp. 109–124, Springer, 1980.

[10] C. Strohm (=Löh). *The Proportionality Principle of Simplicial Volume*, diploma thesis, WWU Münster, 2004. http://arxiv.org/abs/math.AT/0504106

[11] W. P. Thurston. *The geometry and topology of 3-manifolds*, mimeographed notes, 1979. Available online at http://www.msri.org/publications/books/gt3m.

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