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Poincaré's conjecture*

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Abstract. We give an overview of the development on work on Poincaré's conjecture in the first half of the twentieth century.

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Poincaré's conjecture states - in modern terms - that every closed 3-manifold with a vanishing fundamental group is homeomorphic to the 3-sphere. There is a generalization of this conjecture for higher-dimensional manifolds, the so-called generalized Poincaré conjecture (formulated for the first time by W. Hurewicz, [5, p.523]).

In his series of papers on Analysis situs (1892-1904), H. Poincaré studied the question of how to characterize 3-manifolds by invariants. To that end he introduced the fundamental group and investigated Betti-numbers and torsion coefficients. In a first step he realized that there are closed 3-manifolds with identical Betti-numbers but with different fundamental groups (cf. the Manifold Atlas page on Poincaré's cube manifolds, [20]). This result was annouced in 1892 and proven in detail in 1895. Motivated by a critique of P. Heegaard, Poincaré introduced the torsion coefficients in his second complement (1902). In this context Poincaré kept the fundamental group out of sight. But in his last paper - the fifth complement (1904) - he constructed an example of a closed manifold with vanishing first Betti-number, without torsion coefficent but with a fundamental group which he proved to be non-trivial (cf. the Manifold Atlas page on Poincaré's homology sphere, [21]). So the obvious question was: Is the fundamental group strong enough to distinguish 3-manifolds? At the very end of the fifth complement he wrote: "Is it possible that the fundamental group of V is reduced to the identical substitution whereas V is not simply connected?" [13, p.498] - please note that "simply connected" means here "homeomorphic to the sphere". And he added: "But this question would us lead astray." [13, p.498]. So Poincaré didn't formulate a conjecture - there are no indications whether or not he thought the answer to his question should be "yes" or "no". This question simply marks a point in the development of Poincaré's thoughts. Because he frequently used to formulate such questions in his papers - they had often the form of an "inner dialogue" - it is not clear how important the question was in Poincaré's eyes. For the rest of his life Poincaré (+1912) came back neither to his question nor to investigations of the type just described. The analogous question for surfaces - that is the two-dimensional case - was answered in Poincaré's eyes by the classification of closed surfaces worked out in the second half of the 19th century by

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several mathematicians (cf. the Manifold Atlas page on surfaces) - Poincaré himself favoured the results on automorphic functions in this context.

In 1919 J. W. Alexander ([1]) showed, following an argument way sketched by H. Tietze in 1908, that there are closed 3-manifolds with isomorphic fundamental groups that are not homeomorphic (cf. the Manifold Atlas page on Lens spaces in dimension three: a history, [19]). Since the fundamental group in question was not trivial Poincaré's conjecture wasn't touched by it directly. But *the interest of Poincaré's* conjecture grew because now it had become clear that it was an exceptional case.

During the 1920s Poincaré's conjecture became a well known problem. In 1923, B. Kerékjártó wrote in his textbook on the topology of surfaces: "A conjecture by Poincaré states the converse: every closed three-dimensional manifold with a fundamental group reduced to the identy is homeomorphic to the surface of the four-dimensional ball." ([6, p.273]). This seems to be the first place where the misleading term "conjecture" is used. H. Kneser wrote: "One of the most important and obvious questions is whether or not the spherical space is the only simply connected manifold." ([7, p.128]). Four years later he also used the term "well known conjecture" ([8, p.257]). A prominent place was given to Poincaré's conjecture in the "Lehrbuch der Topologie" written by H. Seifert and W. Threlfall: "The 3-sphere is therefore obviously not characterized by its homology-groups. That the fundamental group is enough to characterize it is the content of the until today unproven "Poincaré conjecture"." ([15, p.218]). Beginning with the 1920s there were many references to Poincaré's question or problem; later the term conjecture became predominant. In 1931 the Russian-Austrian topologist F. Frankl published a sort of survey article on the state of the art concerning "Poincaré's question". He discussed several ways of attacking the problem including the equivalent group theoretic formulation. Frankl commented: "In this note I summarize some results produced by failed attempts to solve the problem of homeomorphy for the three-dimensional sphere. They illustrate the great difficulty of this problem." ([2, p.357]).

A first restricted solution was found by H. Seifert in 1932, in the context of his theory of fibered spaces (in modern terms, these are Seifert-fibered spaces, which are 3-manifolds admitting certain S^1-actions). He proved that Poincaré's conjecture is true for 3-manifolds with the structure of a (Seifert-)fibered space. This is the consequence of the more general fact that Seifert-fibered spaces with trivial homology (Poincaré-spaces in Seifert's terminology) are determined by their fundamental group up to homeomorphism ([16, p.197] - for later results on this aspect cf. [3, p.115]). Seifert's work can be seen as the beginning of the geometrization program sketched by W. Thurston in the 1970s. In 1934 J. H. C. Whitehead published a flawed proof of Poincaré's conjecture, correcting it by providing a counter-example to its central theorem the year after. With Whitehead's publication the importance of Poincaré's conjecture became obvious. It is remarkable that his article started with a historical introduction underlining the importance of the problem: Important problems have a history. Milnor commented on Whitehead's failed proof: "Throughout his life, Whitehead retained a deep interest in the very difficult problems which center around the Poincaré conjecture. ... Perhaps this experience [publishing the false demonstration] contributed towards the extreme conscientiousness which marks his later work." ([10, XXIII]). Another false demonstration was published in 1958 by K. Koseki ([9]). In 1986 Rourke and Stewart announced a proof for Poincaré's conjecture which was never published because it soon became clear that the proof was incorrect ([14]).

After the Second World War a lot of research in geometric topology was concerned in a more or less direct way with Poincaré's conjecture (a survey is provided by J. Milnor in [11]). Surprisingly it became obvious that the solution of the generalized Poincaré conjecture in dimensions greater than three was easier than that of the original three-dimensional version. In the year 2000 the Clay Mathematics Institute selected the original Poincaré conjecture as one of its Millenium problems. The definitive solution of Poincaré's conjecture was given by G. Perelman in 2003.

The story of Poincaré's conjecture illustrates nicely the importance of problems (or conjectures) in the development of mathematics as it was presented by Hilbert in his famous talk at Paris (1900): "Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts. It should be to us a guide post on the mazy paths to hidden truths, and ultimately a remainder of our pleasure in the successful solution." ([4, p.438]).

For further related information see also [12], [17], [18], [22], [23] and [24].

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