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Orientation of manifolds in generalized cohomology theories - definition*

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1. Preliminaries

One of classical definitions of orientability of a closed connected manifold M is the existence of the fundamental class $[M] \in H_n(M)$. It is clear that this definition is very suitable to generalize it to generalize (co)homology theories, and this generalization turns out to be highly productive and fruitful.

For the definition of spectra, ring spectra, etc, see [6].

For definitions of generalized (co)homology and their relation to spectra see [6].

The sign $\cong$ denotes an isomorphism of groups or homeomorphism of spaces.

I reserve the term "classical orientation" for orientation in ordinary (co)homology, see e.g. [4].

We denote the ith Stiefel-Whitney class by w_i.

2. Basic definition

Let M be a topological n-dimensional manifold, possibly with boundary. Consider a point $m \in M \setminus \partial M$ and an open disk neighborhood U of m. Let $\varepsilon = \varepsilon^{m,U} : (M, \partial M) \to (S^n, *)$ be the map that collapses the complement of U.

Let E be a commutative ring spectrum, and let $s_n = s_n^E \in E_n(S^n, *)$ be the image of $1 \in \pi_0(E)$ under the isomorphism

$$\pi_0(E) = \widetilde{E}_0(S^0) \cong \widetilde{E}_n(S^n) = E_n(S^n, *).$$

Definition 2.1. Let M be a compact topological n-dimensional manifold (not necessarily connected). An element $[M, \partial M] = [M, \partial M]_E \in E_n(M, \partial M)$ is called an *orientation of M with respect to E*, or, briefly, an *E-orientation* of M, if $\varepsilon_*^{m,U}[M, \partial M] = \pm s_n \in E_n(S^n, *)$ for every m and every disk neighborhood U of m.

Note that a non-connected M is E-orientable iff all its components are.

A manifold with a fixed E-orientation is called E-oriented, and a manifold which admits an E-orientation is called E-orientable. So, an E-oriented manifold is in fact a pair $(M, [M]_E)$.

It follows from the classical orientability that a classically oriented manifold is $H\mathbb{Z}$-orientable, see [4]. Conversely, if a connected manifold is $H\mathbb{Z}$-orientable then $H_n(M, \partial M)) = \mathbb{Z}$ (indeed, we know that either $H_n(M, \partial M) = \mathbb{Z}$ or $H_n(M, \partial M) = 0$, but the second case is impossible because $\varepsilon_* : H_n(M, \partial M) \to H_n(S^n, *)$ must be surjective). Hence, a connected manifold M is $H\mathbb{Z}$-orientable iff $H_n(M, \partial M) = \mathbb{Z}$,

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i.e., iff M is classically orientable. Thus, for arbitrary (not necessarily connected) M is $H\mathbb{Z}$-orientable iff M is classically orientable

Note that s_n is a canonical E-orientation of the sphere S^n.

The following proposition holds because, for every two pairs (m, U) and (m', U') with M connected, the maps $\varepsilon^{m,U}$ and $\varepsilon^{m',U'}$ are homotopic.

Proposition 2.2. *Let M be a connected manifold, and let U_0 be an open disk neighborhood of a point $m_0 \in M \setminus \partial M$. If an element $[M, \partial M] \in E_n(M, \partial M)$ is such that $\varepsilon_*^{m_0,U_0}[M, \partial M] = \pm s_n$, then $[M, \partial M]$ is an E-orientation of M.*

For the proof, see [6, Proposition V.2.2].

3. Number of orientations

Let M be a connected manifold. Let u be and E-orientation of M with $\varepsilon_*(u) = s_n$. Consider another E-orientation u' with $\varepsilon u' = s_n$. Then $\varepsilon_*(u - u') = 0$, and so $u - u' \in \operatorname{Ker}(\varepsilon_* : E_n(M, \partial M) \to E_n(S^n, *))$. Conversely, if $\alpha \in \operatorname{Ker}(\varepsilon_* : E_n(M, \partial M) \to E_n(S^n, *))$ and v is an E-orientation of M then $v + \alpha$ is an E-orientation of M because $\varepsilon_*(u + \alpha) = \varepsilon_*(u)$.

Furthermore, if u is an E-orientation of M with $\varepsilon_*(u) = s_n$ then $-u$ is an E-orientation of M with $\varepsilon_*(-u) = -s_n$,

Thus, if M is a connected E-oriented manifold, then there is a bijection between the set of all E-orientations of M and the set

$$\pm u + \operatorname{Ker}(\varepsilon_* : E_n(M, \partial M) \to E_n(S^n, *)) \subset E_n(M, \partial M),$$

where u is any E-orientation of M.

4. Relation to normal and tangent bundles

Classical orientability of a smooth manifold M is equivalent to the existence of a Thom class of the tangent (or normal) bundle of M, see [4, Theorem 7.1]. The similar claim holds for generalized (co)homology.

Given a vector n-dimensional bundle ξ over a compact space X, consider the Thom space $T\xi$, the one-point compactification of the total space of ξ. Then for every $x \in X$ the inclusion of fiber $\mathbb{R}^n_x = \mathbb{R}^n$ to the total space of ξ yields an inclusion $i_x : S^n = S^x_n \to T\xi$, where S^n_x is the one-point compactification of $\mathbb{R}^n_x$. Now, given a ring spectrum E, note the canonical isomorphism $\widetilde{E}_n(S^n) \cong \widetilde{E}^n(S^n)$ and denote by $s^n \in \widetilde{E}^n(S^n)$ the image of s_n under this isomorphism.

Definition 4.1. A *Thom-Dold class* of ξ with respect to E (on a E-orientation of ξ) is a class $U = U_\xi$ such that $i_x^* U = \pm s^n$ for all $x \in X$.

Theorem 4.2. *A (smooth) manifold M is E-orientable if and only if the tangent (or normal) bundle of M is E-orientable. Moreover, E-orientations of M are in a bijective correspondence with E-orientations of (stable) normal bundle of M.*

For the proof, see [6, Theorem V.2.4 and Corollary V.2.6].

Furthermore, Theorem 4.2 holds for *topological* manifolds as well, if we are careful with the concept of Thom spaces and their normal bundles for topological manifolds, see [6, Definitions IV.5.1 and IV.7.12]. To apply the theory to microbundles, use [6, Theorem IV.7.7].

5. PRODUCTS

Here we show that the product $M \times N$ of two E-oriented manifolds M^m and N^n admits a canonical E-orientation. For sake of simplicity, assume M and N to be closed. Consider two collapsing maps $\varepsilon_M : M \to S^m$ and $\varepsilon_n : N \to S^n$ and form the map

$$M \times N \xrightarrow{\varepsilon_M \times \varepsilon_N} S^m \times S^n \to S^m \wedge S^n = S^{m+n}.$$

It is easy to see that this composition is (homotopic to) $\varepsilon = \varepsilon_{M\times N}$.

Now, let $[M]$ and $[N]$ be E-orientations of M and N, respectively. Consider the commutative diagram

$$\begin{tikzcd}
E_m(M) \otimes E_n(N) \arrow[r, "\mu"] \arrow[d] & E_{m+n}(M \times N) \arrow[r, "\varepsilon_*"] \arrow[d, "(\varepsilon_M)_* \otimes (\varepsilon_N)_*"] & E_{m+n}(S^{m+n}) \arrow[d, "="] \\
E_m(S^m) \otimes E_n(S^n) \arrow[r, "\mu'"] & E_{m+n}(S^m \times S^n) \arrow[r] & E_{m+n}(S^{m+n})
\end{tikzcd}$$

where μ, μ' are given by the ring structure on E. Because of the commutativity of the above diagram, we see that $\varepsilon_*(\mu([M] \otimes [N])) = \pm s_{m+n}$. Thus $[M] \otimes [N]$ is an E-orientation.

It is also worthy to note that if M and $M \times N$ are E-orientable then N is, cf. [6, V.1.10(ii)].

6. POINCARÉ DUALITY

Let F be an E-module spectrum. Given a closed E-oriented manifold $(M, [M]_E)$, consider the homomorphism

$$\frown [M]_E : F^i(M) \to F_{n-i}(M)$$

where $E^i(X) \frown F_j(X) \to F_{j-i}(X)$ is the cap product.

It turns out to be that $\frown [M]_E$ is an isomorphism. This is called *Poincaré duality* and is frequently denoted by $P_M : F^i(M) \to F_{n-i}(M)$.

The Poincaré duality isomorphism admits the following alternative description:

$$P = P_{[M]_E} : F^i(M) \xrightarrow{\varphi} F^i(T\nu) \cong \widetilde{F}_{n-i}(M^+) = F_{n-i}(M).$$

Here $T\nu$ is the Thom spectrum of the stable normal bundle ν of M, and φ is the Thom-Dold isomorphism given by an E-orientation (Thom-Dold class) U of ν, which, in turn, is given by the E-orientation $[M]_E$ of M according to Theorem 4.2.

For the proofs of the statements in this section, see [6, Theorem 2.9].

7. TRANSFER

Definition 7.1. Let F be a module spectrum over a ring spectrum E. Let $f : M^m \to N^n$ be a map of closed manifolds.

Suppose that both M, N are E-oriented, and let P_M, P_N be the Poincaré duality isomorphisms, respectively. We define the *transfers* (other names: *Umkehrs*, *Gysin homomorphisms*)

$$f^! : F^i(M) \to F^{n-m+i}(N), \qquad f_! : F_i(N) \to F_{m-n+i}(M)$$

to be the compositions

$$f^! : F^i(M) \cong F_{m-i}(M) \xrightarrow{f_*} F_{m-i}(N) \cong F^{n-m+i}(M),\ f^! = P^{-1}_{[N]} f_* P_{[M]},$$

$$f_! : F_i(N) \cong F^{n-i}(N) \xrightarrow{f^*} F^{n-i}(M) \cong F_{m-n+i}(N),\ f_! = P_{[M]} f^* P^{-1}_{[N]}.$$

The reader can find many good properties of transfers in Dold [2], Dyer [3], Rudyak [6].

If $f : M^n \to N^n$ is a map of closed $H\mathbb{Z}$-oriented manifolds then

$$f_* f_!(x) = (\deg f)x$$

for every $x \in H_*(N)$. In particular, if $\deg f = 1$ then $f_* : H_*(M) \to H_*(N)$ is epic. Similarly, $f^* : H^*(N) \to H^*(M)$ is a monomorphism if $\deg f = 1$. Theorem 7.2 below generalizes this fact.

Theorem 7.2 ([6, Lemma V.2.12 and Theorem V.2.14]). *Let E be a ring spectrum. Let $f : M^n \to N^n$ be a map of degree ±1 of closed $H\mathbb{Z}$-orientable manifolds. If $[M]$ is an E-orientation of M then $f_*[M]$ is an E-orientation of N. In particular, the manifold N is E-orientable if M is. Moreover, in this case $f^* : F^*(N) \to F^*(M)$ is monic and $f_* : F_*(M) \to F_*(N)$ is epic for every E-module spectrum F.*

8. EXAMPLES

Here we list several examples.

(a) An ordinary (co)homology modulo 2. Represented by the Eilenberg-MacLane spectrum $H\mathbb{Z}/2$. Every manifold is $H\mathbb{Z}/2$-orientable; for M connected the orientation is given be modulo 2 fundamental class. see [2]. Vice versa, if a ring spectrum E is such that every manifold is E-orientable, then E is a graded Eilenberg-MacLane spectrum and $2\pi_*(E) = 0$.

(b) An ordinary (co)homology. Represented by the Eilenberg–MacLane spectrum $H\mathbb{Z}$. By Theorem 4.2 and [6, IV.5.8(ii)], classical orientability is just $H\mathbb{Z}$-orientability. In particular, a smooth manifold is $H\mathbb{Z}$-orientable iff the structure group of its normal and/or tangent bundle can be reduced to SO. Furthermore, $H\mathbb{Z}$-orientability of a manifold M is equivalent to the equality $w_1(M) = 0$.

(c) KO-theory. Atiyah-Bott-Shapiro [1] proved that a smooth manifold M is KO-orientable if and only if it admits a Spin-structure. This holds, in turn, iff $w_1(M) = 0 = w_2(M$. This condition is purely homotopic and can be formulated for every topological manifold (in fact, for Poincaré spaces) in view the equality $w_i(M)U = Sq^i(U)$ where U is the modulo 2 Thom class of the tangent bundle.

The equality $w_1(M) = 0 = w_2(M)$ is necessary for KO-orientability of topological manifolds, but it is not sufficient for KO-orientability even of piecewise linear manifolds, see [6, Ch. VI]. One the other hand, Sullivan proved that every topological manifold is $KO[1/2]$-orientable, see Madsen-Milgram [5] for a good proof. Here $KO[1/2]$ is the $\mathbb{Z}[1/2]$-localized KO-theory.

Note that complex manifold are E-oriented for all E from (a,b,c) (but not (d, e) below).

(d) Complex K-theory. The complexification $C : BO_n \to BU_n$ induces a ring morphism $KO \to K$. So, every KO-orientable manifold is K-orientable.

Atiyah-Bott-Shapiro [1] proved that a smooth manifold M is K-orientable iff it admits a $\mathrm{Spin}^{\mathbb{C}}$-structure. The last condition is equivalent to the purely homotopic conditions $w_1(M) = 0 = \delta w_2(M)$, where δ is the connecting homomorphism in the Bockstein exact sequence

$$\cdots \to H^*(X) \xrightarrow{2} H^*(X) \xrightarrow{\mathrm{mod}\,2} H^*(X;\mathbb{Z}/2) \xrightarrow{\delta} H^*(X) \to \dots .$$

This condition is necessary for K-orientability of manifolds, but it is not sufficient for K-orientability of piecewise linearly (and hence topological) manifolds, see [6, Ch. VI]. On the other hand, every classically oriented topological manifold is $K[1/2]$-orientable in view of Sullivan's result mentioned in example (c).

(e) Stable (co)homotopy groups, or frames (co)bordism theory. Represented by the spectrum S. Because of Theorem 4.2, a manifold M^n is orientable with respect to the sphere spectrum S iff its tangent bundle τM has trivial stable fiber homotopy type, i.e., iff there exists N such that $\tau \oplus \theta^N$ is equivalent to θ^{N+n} where θ^k is a trivial k-dimensional bundle. In particular, we have the following necessary (but not sufficient) condition: $w_i(M) = 0$ for all i.

Note that S-orientability implies KO-orientability implies K-orientability implies $H\mathbb{Z}$-orientability implies $H\mathbb{Z}/2$-orientability. Furthermore, any S-orientable manifold is E-orientable for every ring spectrum E, cf. [6, I.1.6]. So, (a) and (e) appear as two extremal cases.

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