Orientation covering - definition*

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1. Construction

Let $M$ be a $n$-dimensional topological manifold. We construct an oriented manifold $\hat{M}$ and a 2-fold covering $p : \hat{M} \to M$ called the orientation covering. The non-trivial deck transformation of this covering is orientation reversing. As a set $\hat{M}$ is the set of pairs $(x, o_x)$, where $o_x$ is a local orientation of $M$ at $x$ given by a generator of the infinite cyclic group $H_n(M, M - x; \mathbb{Z})$. The map $p$ assigns $x$ to $(x, o_x)$. Since there are precisely two local orientations, the fibres of this map have cardinality 2.

Next we define a topology on this set. Let $\varphi : U \to V \subset \mathbb{R}^n$ be a chart of $M$. We orient $\mathbb{R}^n$ by the standard orientation given by the standard basis $e_1, e_2, \ldots, e_n$, from which we define a continuous local orientation by identifying the tangent space with $\mathbb{R}^n$. Since for a smooth manifold a tangential orientation defines a homological orientation, this also gives a homological orientation: see [2, §3]. We call the standard local orientation at $x \in \mathbb{R}^n$ by $sto_x$. Using the chart we transport this standard orientation to $U$ by the induced map on homology. The local orientations given by this orientation of $U$ determine a subset of $\hat{M}$, which we require to be open. Doing the same starting with the non-standard orientation of $\mathbb{R}^n$ we obtain another subset, which we also call open. We give $\hat{M}$ the topology generated by these open subsets, where we vary about all charts. By construction each of these open subsets is homeomorphic to an open subset of $\mathbb{R}^n$, and so we obtain an atlas of $\hat{M}$. The map $p$ is by construction a 2-fold covering. By construction $\hat{M}$ is oriented in a tautological way and the non-trivial deck transformation of the covering is orientation reversing.

Thus we have constructed a 2-fold covering of $M$ by an oriented manifold $\hat{M}$, which is smooth, if $M$ is smooth. This covering is called the orientation covering.

If $M$ is smooth one can use the local tangential orientation of $T_x M$ instead of the homological orientation to construct the orientation covering (for the equivalence of these data see the Manifold Atlas page Orientation of manifolds; [2, §6]). Since a countable covering of a smooth manifold has a unique smooth structure such that the projection map is a local diffeomorphism, in the smooth case $\hat{M}$ is a smooth manifold and $p$ a local diffeomorphism.

For more information and a discussion placing the orientation covering in a wider setting, see [1, VIII §2].

*Atlas page: www.map.mpim-bonn.mpg.de/Orientation_covering

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2. Characterization of the orientation covering

One can easily characterize the orientation covering:

**Proposition 2.1.** If \(N\) is an oriented manifold and \(p: N \to M\) is a 2-fold covering with orientation reversing non-trivial deck transformation, then it is isomorphic to the orientation covering.

**Proof.** We have a map \(N \to \hat{M}\) by mapping \(y \in N\) to \((p(y), \text{orientation induced by } p)\). This is an isomorphism of these two coverings. \(\square\)

If \(M\) is orientable, we pick an orientation and see that \(\hat{M}\) is the disjoint union of \(\{(x, o_x)\mid o_x \text{ is the local orientation given by the orientation of } M\}\) and its complement, so it is isomorphic to the trivial covering \(M \times \mathbb{Z}/2\). In turn if the orientation covering is trivial it decomposes \(\hat{M}\) into two open (and thus oriented) subsets homeomorphic to \(M\) and so \(M\) is orientable. Thus we have shown:

**Proposition 2.2.** \(M\) is orientable if and only if the orientation covering is trivial. If \(M\) is connected, \(M\) is non-orientable if and only if \(\hat{M}\) is connected. In particular, any simply-connected manifold is orientable.

3. Relation to the orientation character

We assume now that \(M\) is connected. The orientation character is a homomorphisms \(w: \pi_1(M) \to \{\pm 1\}\), which attaches +1 to a loop \(S^1 \to M\) if and only if the pull back of the orientation covering is trivial. By the classification of coverings this implies that \(w\) is trivial if and only if \(M\) is orientable.

4. Examples

Here are some examples of orientation coverings.

(1) If \(M\) is orientable then \(p: \hat{M} \to M\) is isomorphic to the projection \(M \times \mathbb{Z}/2 \to M\).

(2) If \(n\) is even, \(\mathbb{R}P^n\) is non-orientable and the orientation cover is the canonical projection \(S^n \to \mathbb{R}P^n\). The deck transformation of the orientation covering is the antipodal map on \(S^n\).

(3) The orientation cover of the Klein bottle \(K^2\) is the canonical projection from the 2-torus; \(p: T^2 \to K^2\).

(4) The orientation cover of the open Möbius strip \(M\bar{o}\) is the canonical projection from the cylinder; \(p: S^1 \times \mathbb{R} \to M\bar{o}\).

References


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