Orientation covering - definition*

MATTHIAS KRECK

1. Construction

Let $M$ be a $n$-dimensional topological manifold. We construct an oriented manifold $\hat{M}$ and a 2-fold covering $p : \hat{M} \to M$ called the orientation-reversing covering. The non-trivial deck transformation of this covering is orientation-reversing. As a set $\hat{M}$ is the set of pairs $(x,o_x)$, where $o_x$ is a local orientation of $M$ at $x$ given by a generator of the infinite cyclic group $H_n(M,M-x;\mathbb{Z})$. The map $p$ assigns $x$ to $(x,o_x)$. Since there are precisely two local orientations, the fibres of this map have cardinality 2.

Next we define a topology on this set. Let $\varphi : U \to V \subset \mathbb{R}^n$ be a chart of $M$. We orient $\mathbb{R}^n$ by the standard orientation given by the standard basis $e_1, e_2, \ldots, e_n$, from which we define a a continuous local orientation by identifying the tangent space with $\mathbb{R}^n$. Since for a smooth manifold a tangential orientation defines a homological orientation, this also gives a homological orientation: see [2, §3]. We call the standard local orientation at $x \in \mathbb{R}^n$ by $sta_x$. Using the chart we transport this standard orientation to $U$ by the induced map on homology. The local orientations given by this orientation of $U$ determine a subset of $\hat{M}$, which we require to be open. Doing the same starting with the non-standard orientation of $\mathbb{R}^n$ we obtain another subset, which we also call open. We give $\hat{M}$ the topology generated by these open subsets, where we vary about all charts. By construction each of these open subsets is homeomorphic to an open subset of $\mathbb{R}^n$, and so we obtain an atlas of $\hat{M}$. The map $p$ is by construction a 2-fold covering. By construction $\hat{M}$ is oriented in a tautological way and the non-trivial deck transformation of the covering is orientation reversing.

Thus we have constructed a 2-fold covering of $M$ by an oriented manifold $\hat{M}$, which is smooth, if $M$ is smooth. This covering is called the orientation covering.

If $M$ is smooth one can use the local tangential orientation of $T_xM$ instead of the homological orientation to construct the orientation covering (for the equivalence of these data see the Manifold Atlas page Orientation of manifolds; [2, §6]). Since a countable covering of a smooth manifold has a unique smooth structure such that the projection map is a local diffeomorphism, in the smooth case $\hat{M}$ is a smooth manifold and $p$ a local diffeomorphism.

For more information and a discussion placing the orientation covering in a wider setting, see [1, VIII §2].

*Atlas page :www.map.mpim-bonn.mpg.de/Orientation_covering

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2. Characterization of the orientation covering

One can easily characterize the orientation covering:

**Proposition 2.1.** If \( N \) is an oriented manifold and \( p : N \to M \) is a 2-fold covering with orientation reversing non-trivial deck transformation, then it is isomorphic to the orientation covering.

**Proof.** We have a map \( N \to \hat{M} \) by mapping \( y \in N \) to \((p(y), \text{orientation induced by } p)\). This is an isomorphism of these two coverings. \(\square\)

If \( M \) is orientable, we pick an orientation and see that \( \hat{M} \) is the disjoint union of \( \{(x, o_x) | o_x \text{ is the local orientation given by the orientation of } M\} \) and its complement, so it is isomorphic to the trivial covering \( M \times \mathbb{Z}/2 \). In turn if the orientation covering is trivial it decomposes \( \hat{M} \) into two open (and thus oriented) subsets homeomorphic to \( M \) and so \( M \) is orientable. Thus we have shown:

**Proposition 2.2.** \( M \) is orientable if and only if the orientation covering is trivial. If \( M \) is connected, \( M \) is non-orientable if and only if \( \hat{M} \) is connected. In particular, any simply-connected manifold is orientable.

3. Relation to the orientation character

We assume now that \( M \) is connected. The orientation character is a homomorphisms \( w : \pi_1(M) \to \{\pm 1\} \), which attaches +1 to a loop \( S^1 \to M \) if and only if the pull back of the orientation covering is trivial. By the classification of coverings this implies that \( w \) is trivial if and only if \( M \) is orientable.

4. Examples

Here are some examples of orientation coverings.

1. If \( M \) is orientable then \( \hat{M} \to M \) is isomorphic to the projection \( M \times \mathbb{Z}/2 \to M \).
2. If \( n \) is even, \( \mathbb{R}P^n \) is non-orientable and the orientation cover is the canonical projection \( S^n \to \mathbb{R}P^n \). The deck transformation of the orientation covering is the antipodal map on \( S^n \).
3. The orientation cover of the Klein bottle \( K^2 \) is the canonical projection from the 2-torus; \( p : T^2 \to K^2 \).
4. The orientation cover of the open Möbius strip \( M\hat{o} \) is the canonical projection from the cylinder; \( p : S^1 \times \mathbb{R} \to M\hat{o} \).

References


Matthias Kreck
Endenicher Allee 60
D-53115 Bonn, Germany

E-mail address: kreck@math.uni-bonn.de
Web address: http://him.uni-bonn.de/homepages/prof-dr-matthias-kreck/