Orientation covering - definition*

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1. Construction

Let M be a n-dimensional topological manifold. We construct an oriented manifold $\hat{M}$ and a 2-fold covering $p : \hat{M} \to M$ called the orientation covering. The non-trivial deck transformation of this covering is orientation-reversing. As a set $\hat{M}$ is the set of pairs (x, o_x), where o_x is a local orientation of M at x given by a generator of the infinite cyclic group $H_n(M, M - x; \mathbb{Z})$. The map p assignes x to (x, o_x). Since there are precisely two local orientations, the fibres of this map have cardinality 2.

Next we define a topology on this set. Let $\varphi : U \to V \subset \mathbb{R}^n$ be a chart of M. We orient $\mathbb{R}^n$ by the standard orientation given by the standard basis e_1, e_2, ..., e_n, from which we define a a continuous local orientation by identifying the tangent space with $\mathbb{R}^n$. Since for a smooth manifold a tangential orientation defines a homological orientation, this also gives a homological orientation: see [2, §3]. We call the standard local orientation at $x \in \mathbb{R}^n$ by sto_x. Using the chart we transport this standard orientation to U by the induced map on homology. The local orientations given by this orientation of U determine a subset of $\hat{M}$, which we require to be open. Doing the same starting with the non-standard orientation of $\mathbb{R}^n$ we obtain another subset, which we also call open. We give $\hat{M}$ the topology generated by these open subsets, where we vary about all charts. By construction each of these open subsets is homeomorphic to an open subset of $\mathbb{R}^n$, and so we obtain an atlas of $\hat{M}$. The map p is by construction a 2-fold covering. By construction $\hat{M}$ is oriented in a tautological way and the non-trivial deck transformation of the covering is orientation reversing.

Thus we have constructed a 2-fold covering of M by an oriented manifold $\hat{M}$, which is smooth, if M is smooth. This covering is called the **orientation covering**.

If M is smooth one can use the local tangential orientation of T_xM instead of the homological orientation to construct the orientation covering (for the equivalence of these data see the Manifold Atlas page Orientation of manifolds; [2, §6]). Since a countable covering of a smooth manifold has a unique smooth structure such that the projection map is a local diffeomorphism, in the smooth case $\hat{M}$ is a smooth manifold and p a local diffeomorphism.

For more information and a discussion placing the orientation covering in a wider setting, see [1, VIII §2].

*Atlas page :www.map.mpim-bonn.mpg.de/Orientation_covering

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2. Characterization of the orientation covering

One can easily characterize the orientation covering:

Proposition 2.1. *If N is an oriented manifold and $p : N \to M$ is a 2-fold covering with orientation reversing non-trivial deck transformation, then it is isomorphic to the orientation covering.*

Proof. We have a map $N \to \hat{M}$ by mapping $y \in N$ to $(p(y), \textit{orientation induced by } p)$. This is an isomorphism of these two coverings. □

If M is orientable, we pick an orientation and see that $\hat{M}$ is the disjoint union of $\{(x, o_x)|\ o_x \textit{ is the local orientation given by the orientation of } M\}$ and its complement, so it is isomorphic to the trivial covering $M \times \mathbb{Z}/2$. In turn if the orientation covering is trivial it decomposes $\hat{M}$ into two open (and thus oriented) subsets homeomorphic to M and so M is orientable. Thus we have shown:

Proposition 2.2. *M is orientable if and only if the orientation covering is trivial. If M is connected, M is non-orientable if and only if $\hat{M}$ is connected. In particular, any simply-connected manifold is orientable.*

3. Relation to the orientation character

We assume now that M is connected. The orientation character is a homomorphims $w : \pi_1(M) \to \{\pm 1\}$, which attaches $+1$ to a loop $S^1 \to M$ if and only if the pull back of the orientation covering is trivial. By the classification of coverings this implies that w is trivial if and only if M is orientable.

4. Examples

Here are some examples of orientation coverings.

(1) If M is orientable then $p\colon \hat{M} \to M$ is isomorphic to the projection $M \times \mathbb{Z}/2 \to M$.
(2) If n is even, $\mathbb{R}P^n$ is non-orientable and the orientation cover is the canonical projection $S^n \to \mathbb{R}P^n$. The deck transformation of the orientation covering is the antipodal map on S^n.
(3) The orientation cover of the Klein bottle K^2 is the canonical projection from the 2-torus; $p\colon T^2 \to K^2$.
(4) The orientation cover of the open Möbius strip Mö is the canonical projection from the cylinder; $p\colon S^1 \times \mathbb{R} \to M$ö.

References

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