Fundamental class - definition*

MATTHIAS KRECK

1. The integral fundamental class

For compact manifolds one can characterize the orientability by the existence of a certain homology class called the fundamental class. The background is the following theorem. Recall that if M is an n-dimensional topological manifold (possibly with boundary), then for each x in the interior of M, one has $H_n(M, M-x;\mathbb{Z}) \cong \mathbb{Z}$, [2, 22.1].

Theorem 1.1. *Let M be an n-dimensional compact topological manifold (possibly with boundary). If M is connected and orientable then for each x in the interior of M, the map induced by the inclusion*

$$H_n(M,\partial M;\mathbb{Z}) \to H_n(M, M-x;\mathbb{Z}),$$

is an isomorphism. In particular, $H_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}$. If M is connected and non-orientable then $H_n(M,\partial M; Z)$ is zero.

Proof. If M is closed then this is part of [1, VIII Corollary 3.4]; see also [2, 22.26]. If M has a boundary ∂M, then the inclusion $\partial M = \partial M \times \{1\} \subset M$ extends to an embedding $\partial M \times I \subset M$ of a collar, where $I = [0,1]$ [3, Proposition 3.42]. Let $M_0 = \text{cl.}(M - \partial M \times I)$, so that $M = M_0 \cup_{\partial M} (\partial M \times I)$. By excision

$$H_n(M,\partial M;\mathbb{Z}) \cong H_n(M,\partial M \times I;\mathbb{Z}) \cong H_n(M - \partial M \times \{1\}, \partial M \times [0,1);\mathbb{Z}).$$

Now apply [1, VIII Corollary 3.4] to the open manifold $X := M - \partial M \times \{1\}$ and the closed subset $M_0 \subset X$. □

Theorem 1.1 implies that a connected compact manifold M is orientable if and only if

$$H_n(M,\partial M;\mathbb{Z}) \cong \mathbb{Z}.$$

A choice of a generator is then called a **fundamental class** $[M,\partial M] \in H_n(M,\partial M;\mathbb{Z})$ for M. The fundamental class determines by the isomorphism above a continuous choice of local orientations and in turn the fundamental class is determined by a homological orientation of M. In other words a connected compact manifold together with the choice of a fundamental class $[M,\partial M]$ is the same as an oriented manifold. If M is not connected, then M is orientable if and only all components are orientable. If the components are oriented the fundamental classes of the components give the **fundamental class of** M under the isomorphism which decomposes the homology groups into the homology groups of the components. Thus for oriented manifolds again one has a fundamental class which corresponds to a orientation as in the connected case. The construction of the fundamental class of an oriented closed

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manifold is done inductively over an atlas (similarly for manifolds with boundary). Namely one has the following generalization of Theorem 1:

Theorem 1.2 ([2, 22.24]). *Let M be a connected oriented n-dimensional manifold. Then for each compact subset $K \subset M$ there is a class $[M]_K \in H_n(M, M-K)$ such that the following hold.*

(1) *If $K \subset K'$ is another compact subset, then $[M]'_K$ maps to M_K under the map induced by the inclusion.*
(2) *For each $x \in M$ the class $[M]_x$ is the local orientation of M.*
(3) *The classes M_K are uniquely characterized by these properties.*

Using this one can use the Mayer-Vietoris sequence to 'glue' together the local orientations inductively over a finite oriented atlas together to construct M_K. The inductive construction is rather indirect. If one defines the homology of a space X as the bordism classes of certain stratified spaces $\mathcal{S}$ together with a continuous map to X, e.g. stratifolds, then the the fundamental class is easy to obtain, it is a tautology. Then the fundamental class of a closed manifold is the bordism class represented by the identity map

$$id : M \to M.$$

For this see [4, Chapter 7, Section 1].

2. The $\mathbb{Z}/2$-fundamental class

For all n-dimensional connected compact manifolds - even if they are not orientable - one has

$$H_n(M, \partial M; \mathbb{Z}/2) = \mathbb{Z}/2,$$

and one calls the non-trivial element the $\mathbb{Z}/2$-fundamental class [1, VIII Definition 4.1]. As for the integral fundamental class (if M is oriented) one gets from these classes the $\mathbb{Z}/2$**-fundamental class** of a non-connected compact manifold. Also one has a generalization of Theorem 1.2 to non-compact connected manifolds, i.e. for each compact subset K one has $H_n(M, M-K; \mathbb{Z}/2) \cong \mathbb{Z}/2)$ and for $K \subset K'$ the map induced by the inclusion is an ismorphism

$$H_n(M, M-K'; \mathbb{Z}/2) \to H_n(N, M-K; \mathbb{Z}/2).$$

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MATTHIAS KRECK
ENDENICHER ALLEE 60
D-53115 BONN, GERMANY

E-mail address: kreck@math.uni-bonn.de
Web address: http://him.uni-bonn.de/homepages/prof-dr-matthias-kreck/