Bulletin of the Manifold Atlas - definition (2014)

Wu class - definition*

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1. Introduction

The Wu class of a manifold M is a characteristic class allowing a computation of the Stiefel-Whitney classes of M by knowing only $H^*(M;\mathbb{Z}/2)$ and the action of the Steenrod squares.

2. Definition

Let M be a closed topological n-manifold, $[M] \in H_n(M;\mathbb{Z}/2)$ its fundamental class, Sq^k the k-th Steenrod square and

$$\langle\ ,\ \rangle : H^i(M;\mathbb{Z}/2) \times H_i(M;\mathbb{Z}/2) \longrightarrow \mathbb{Z}/2$$

the usual Kronecker pairing. This pairing, together with the Poincarè duality isomorphism $a \mapsto a \cap [M]$, induces isomorphisms

$$\mathrm{Hom}(H^{n-k}(M;\mathbb{Z}/2),\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2) \cong H^k(M;\mathbb{Z}/2),$$

under which the homomorphism $x \mapsto \left\langle Sq^k(x),[M]\right\rangle$ from $H^{n-k}(M;\mathbb{Z}/2)$ to $\mathbb{Z}/2$ corresponds to a well defined cohomology class $v_k \in H^k(M;\mathbb{Z}/2)$. This cohomology class is called the k-th Wu class of M ([4, §11]). We may rewrite its definition equivalently as an identity

$$\langle v_k \cup x,[M]\rangle = \left\langle Sq^k(x),[M]\right\rangle \qquad \text{for all } x \in H^{n-k}(M;\mathbb{Z}/2). \tag{1}$$

Define the total Wu class

$$v \in H^*(M;\mathbb{Z}/2) = H^0(M;\mathbb{Z}/2) \oplus H^1(M;\mathbb{Z}/2) \oplus ... \oplus H^n(M;\mathbb{Z}/2),$$

as the formal sum

$$v := 1 + v_1 + v_2 + ... + v_n.$$

Using the total Steenrod square,

$$Sq := Sq^0 + Sq^1 + Sq^2 + \dots : H^*(M;\mathbb{Z}/2) \longrightarrow H^*(M;\mathbb{Z}/2),$$

equation (1) translates into the following formula

$$\langle v \cup x,[M]\rangle = \langle Sq(x),[M]\rangle \qquad \text{for all } x \in H^*(M;\mathbb{Z}/2), \tag{2}$$

which may also be used as a definition of the total Wu class of M.

From the definition it is clear that the Wu class is defined even for a Poincarè complex M.

*Atlas page: www.map.mpim-bonn.mpg.de/Wu_class

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3. Relation to Stiefel-Whitney classes

From now on all manifolds are supposed to be smooth. The following theorem of Wu Wen-Tsun ([7]) allows a computation of the Stiefel-Whitney classes $w_i(M)$ of M using only $H^*(M;\mathbb{Z}/2)$ and the action of the Steenrod squares:

Theorem 1. The total Stiefel-Whitney class of M,

$$w = w(M) := 1 + w_1(M) + w_2(M) + ... + w_n(M),$$

is given by

$$w(M) = Sq(v),$$

or equivalently

$$w_k(M) = \sum_{i=0} Sq^i(v_{k-i}).$$

For a proof see [4, §11]. Since Sq is a ring automorphism of

$$H^{**}(X;\mathbb{Z}/2) := \prod_{i\geq 0} H^i(X;\mathbb{Z}/2),$$

Sq^{-1} is defined on $H^{**}(M;\mathbb{Z}/2) = H^*(M;\mathbb{Z}/2)$ and we may write

$$v = Sq^{-1}(w(M)).$$

The formula $w(M) = Sq(v)$ may be used to extend the definition of the Stiefel-Whitney classes to the class of topological manifolds.

4. An example

The following example is taken from [4, §11]. If $H^*(M;\mathbb{Z}/2)$ is of the form $\mathbb{Z}/2[x]/(x^{dm+1})$, where $x \in H^d(X;\mathbb{Z}/2)$, $d \geq 1$, $n = d \cdot m$, for example if $M = \mathbb{C}P^m$, then

$$v = (1 + x + x^2 + x^4 + \dots)^{m+1} \quad \text{and} \quad w(M) = (1+x)^{m+1}$$

with

$$Sq(x) = x + x^2 \quad \text{and} \quad Sq^{-1}(x) = x + x^2 + x^4 + \dots .$$

5. A generalization

The following example is taken from [1]. Let λ be a natural ring automorphism of $H^{**}(X;\mathbb{Z}/2)$ and Φ_ξ the Thom isomorphism of a real vector bundle ξ on X. Define

$$(3)\quad \underline{\lambda} = \underline{\lambda}(\xi) := \Phi_\xi^{-1} \circ \lambda \circ \Phi_\xi(1) \ \text{ and } \ \mathrm{Wu}(\lambda,\xi) := \lambda^{-1} \circ \underline{\lambda} = \lambda^{-1} \circ \Phi_\xi^{-1} \circ \lambda \circ \Phi_\xi(1).$$

If $\lambda = Sq$, then $\underline{\lambda} = w$ is the total Stiefel-Whitney classes $w(\xi)$ of ξ ([4, §8]) and with $\xi = \tau M$, the tangent bundle of $X = M$, we have $\mathrm{Wu}(Sq, \tau M) = v$, the total Wu class of M.

In general $\xi \mapsto \underline{\lambda}(\xi)$ and $\xi \mapsto \mathrm{Wu}(\lambda,\xi)$ define multiplicative characteristic classes, translating Whitney sum into cup product, i.e. they satisfy a Whitney product type formula

$$\underline{\lambda}(\xi \oplus \eta) = \underline{\lambda}(\xi) \cup \underline{\lambda}(\eta) \ \text{ and } \ \mathrm{Wu}(\lambda, \xi\oplus\eta) = \mathrm{Wu}(\lambda,\xi) \cup \mathrm{Wu}(\lambda,\eta).$$

Such a characteristic class is determined by a power series $f(x) \in \mathbb{Z}/2[[x]]$, which is given by its value on the universal line bundle. The generalized Wu class $\mathrm{Wu}(\lambda,\xi)$ is defined as a commutator class, thus measuring how λ and Φ_ξ commute. This is similar to the situation considered in the (differential) Riemann-Roch formulas, in which the interaction between the Chern character and the Thom isomorphism in K-Theory and rational cohomology is formulated. This relation is more than only formal: Let T_i be the i-th Todd polynomial, then $2^i \cdot T_i$ is a rational polynomial with denominators prime to 2, hence its reduction to mod 2 cohomology is well defined. Then Atiyah and Hirzebruch proved:

Theorem 2 ([1]).
$$\mathrm{Wu}(Sq,\xi) = \sum_{i\geq 0} 2^i \cdot T_i(w_1(\xi), w_2(\xi), \dots, w_i(\xi)) \quad \text{in} \quad H^{**}(X;\mathbb{Z}/2).$$

The proof is by comparing the power series belonging to the multiplicative characteristic classes on both sides of the equation, which turn out to be $x/Sq^{-1}(x) = 1 + \sum_{j\geq 0} x^{2^j}$.

For a continuous map $f : M \to N$ between closed differentiable manifolds the analogue of the Riemann-Roch formula is
$$f_!(\lambda(x) \cup \mathrm{Wu}(\lambda^{-1}, \tau M)) = \lambda(f_!(x)) \cup \mathrm{Wu}(\lambda^{-1}, \tau N).$$

Here $f_!$ is the Umkehr map of f defined by f_* via Poincarè duality. In the case $f : M \to *$, this reduces to $\langle \mathrm{Wu}(\lambda, \tau M) \cup x, [M] \rangle = \langle \lambda(x), [M] \rangle$, generalizing (2).

6. Applications

(1) The definition of the total Wu class v and $w = Sq(v)$ show, that the Stiefel-Whitney classes of a smooth manifold are invariants of its homotopy type.
(2) Since the Stiefel-Whitney classes of a closed n-manifold determine its un-oriented bordism class [6, Théorèm IV.10], a corollary of (1) is: Homotopy equivalent manifolds are un-oriented bordant.
(3) Inserting the Stiefel-Whitney classes of M for x in
$$\langle v \cup x, [M] \rangle = \langle Sq(x), [M] \rangle ,$$
and using $v = Sq^{-1}(w)$ one gets relations between Stiefel-Whitney numbers of n-manifolds. It is a result of Dold ([2]) that all relations between Stiefel-Whitney numbers of n-manifolds are obtained in this way.

4. Conditions on the Wu classes v_s for nonbounding manifolds are given in [5].
5. For an appearance of the Wu class in surgery theory see [3, Ch. 4].

Remarks:
1. Most of the above has analogues for odd primes, e.g. see [1].
2. Not directly related to the Wu class is Wu's explicit formula for the action of Steenrod squares on the Stiefel-Whitney classes of a vector bundle ξ (see [4, §8]):
$$Sq^k(w_m(\xi)) = w_k \cup w_m + \binom{k-m}{1} w_{k-1} \cup w_{m+1} + \cdots + \binom{k-m}{k} w_0 \cup w_{m+k}$$
where $\binom{x}{i} = x(x-1)\dots(x-i+1)/i!$.

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