Bulletin of the Manifold Atlas - definition (2014)

Hilbert manifold - definition*

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1. Introduction and Definition

Even if one is interested only in finite-dimensional manifolds, the need for infinite-dimensional manifolds sometimes arises. For example, one approach to study closed geodesics on a manifold is to use Morse theory on its (free) loop space; while for some purposes it is enough to work with finite-dimensional approximations, it is helpful for some finer aspects of the theory to use models of the free loop space that are infinite-dimensional manifolds. The use of Morse theory in an infinite-dimensional context is even more important for other (partial) differential equations like those occuring in the theory of minimal surfaces and the Yang-Mills equations. Morse theory for infinite dimensional manifolds was developed by Palais and Smale ([20], [21]).

While there is up to isomorphism only one vector space of every finite dimension, there are many different kinds of infinite-dimensional topological vector spaces one can choose. Modeling spaces on Fréchet spaces gives the notion of Fréchet manifolds, modelling on Banach spaces gives Banach manifolds, modelling on the Hilbert cube (the countably infinite product of intervals) gives Hilbert cube manifolds. We will stick to Hilbert manifolds (which are not directly related to Hilbert cube manifolds).

Definition 1.1. Let H be the (up to isomorphism unique) separable Hilbert space of infinite dimension. Then a *Hilbert manifold* is a separable metrizable space such that every point has a neighborhood that is homeomorphic to an open subset of H.

Some authors have slightly different definitions, leaving out the metrizability or the separability condition. Note that metrizability always implies paracompactness and here also the converse is true. Being metrizable and separable is in this context also equivalent to being second countable and Hausdorff by Uryson's metrization theorem (see also [11, 4(A)]).

Note that every separable Frechet space is homeomorphic to the separable Hilbert space (see [1]). Thus, the structure of a topological Hilbert manifold is not different from that of a topological Frechet manifold; only in the differentiable case differences show up. A $\mathcal{C}^k$-structure for $k = 0, 1, \dots, \infty, \omega$ can be defined as usual as an equivalence class of atlases whose chart transition maps are of class $\mathcal{C}^k$. Here, $\mathcal{C}^\omega$ stands for analytic functions. The tangent bundle of Hilbert manifold can also be defined as usual for $k \geq 1$ and is a Hilbert space bundle with structure group $GL(H)$ with the norm topology (see [17], II.1 and III.2).

A *submanifold* of a Hilbert manifold X is a subset $Y \subset X$ such that for every point $y \in Y$ there is an open neighborhood V of y in X and a homeomorphism

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$\psi : V \to W$ to an open subset $W \subset H$ such that $\psi(V \cap Y) = W \cap U$ for U a *closed* linear subspace of H.

2. Properties

2.1. **Basic Differential Topology.** Many basic theorems of differential topology carry over from the finite dimensional situation to the Hilbert (and even Banach) setting with little change. For example, every smooth submanifold of a smooth Hilbert manifold has a tubular neighborhood, unique up to isotopy (see [17] IV.5-6 and also [23] for the non-closed case.). Also, every Hilbert manifold can be embedded as a closed submanifold into the standard Hilbert space ([15]). However, in statements involving maps between manifolds, one often has to restrict consideration to *Fredholm maps*, i.e. maps whose differential at every point has closed image and finite-dimensional kernel and cokernel. The reason for this is that Sard's lemma holds for Fredholm maps, but not in general (see [24] and [3]). The precise statement is:

Theorem 2.1 ([24]). *Let $f : M \to N$ be a smooth Fredholm map between Hilbert manifolds. Then its set of regular values is the intersection of countably many sets with dense interior.*

2.2. **Homotopy Theory.**

Theorem 2.2 ([22], Theorem 5, Theorem 14]). *Every Hilbert manifold is an absolute neighborhood retract and has therefore the homotopy type of a countable, locally finite simplicial complex.*

On the other hand, every countable, locally finite simplicial complex is homotopy equivalent to an open subset of the standard Hilbert space. Thus, the homotopy classification of Hilbert manifolds is equivalent to that of countable, locally finite simplicial complexes or, equivalently, countable CW-complexes (see [6], Section 10).

2.3. **Specialties of Infinite Dimension.** While proofs are often harder in infinite dimensions, some things are true for Hilbert manifolds that could not be hoped for in finite dimensions.

Theorem 2.3 ([16]). *The unitary group and the general linear group of the (real or complex) separable infinite-dimensional Hilbert space are contractible with the norm topology.*

Corollary 2.4. *If X is a paracompact space, then every (real or complex) Hilbert space vector bundle with these structure groups over X is trivial. In particular, every (smooth) Hilbert manifold is parallelizable.*

Theorem 2.5 ([12], [10]). *Every Hilbert manifold X can be embedded onto an open subset of the model Hilbert space.*

Theorem 2.6 ([6], [19]). *Every homotopy equivalence between two smooth Hilbert manifolds is homotopic to a diffeomorphism. In particular every two homotopy equivalent smooth Hilbert manifolds are already diffeomorphic.*

Indeed, Burghelea and Kuiper show this result under the assumption of the existence of a Morse function and Moulis shows the existence of a Morse function on an open subset of the standard Hilbert space.

Theorem 2.7 ([5], [7]). *Every topological Hilbert manifolds possesses a (unique) smooth structure. Every homeomorphism of a differentiable Hilbert manifold is isotopic (through homeomorphisms) to a diffeomorphisms. Moreover every two homotopic diffeomorphisms are isotopic (through diffeomorphisms).*

Thus, the category of topological Hilbert manifolds with homeomorphisms up to isotopy and the category of smooth Hilbert manifolds with diffeomorphisms up to isotopy are equivalent.

The situation is different for complex analytic structures. These always exist, but are not unique. Indeed, there are infinitely many nonequivalent complex analytic structures on every Hilbert manifold (see [4]).

Although Sard's Theorem does not hold in full generality, note that we have also the following theorem:

Theorem 2.8 ([2]). *Every continuous map $f : X \to \mathbb{R}^n$ from a Hilbert manifold can be arbitrary closely approximated by a smooth map $g : X \to \mathbb{R}^n$ that has no critical points.*

3. Examples

Example 3.1. Any open subset U of a separable Hilbert space H is a Hilbert manifold with a single global chart given by the inclusion into H. Up to diffeomorphisms, every Hilbert manifold is of this form by [12].

Example 3.2. The unit sphere in a separable Hilbert space is a smooth Hilbert manifold.

Example 3.3. Mapping spaces between manifolds can often be viewed as Hilbert manifolds if one considers only maps of suitable Sobolev class. Set $H^r = W^{2,n}$ to be the Sobolev class of L^2-functions which are k-fold weakly differentiable in L^2. Let now M be an n-dimensional compact smooth manifold, N be an arbitrary smooth finite-dimensional or Hilbert manifold and $Map(M,N)$ be the space of continuous maps with the compact-open topology. Then the subspace $Sob(M,N) \subset Map(M,N)$ of functions of Sobolev type H^r for $r > n/2$ can be given the structure of a smooth Hilbert manifold ([11, 6(D)]). This inclusion is a homotopy equivalence ([11, 6(E)]). In particular, $Sob(M,N)$ is diffeomorphic to any other Hilbert manifold homotopy equivalent to $Map(M,N)$ and therefore its diffeomorphism type depends only on the homotopy type of M and N. Hilbert manifold models for mapping spaces (in particular, free loop spaces) have been used, for example, in the study of closed geodesics ([14], [13]), string topology ([8], [18]) and fluid dynamics ([9]).

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