Contact manifold - definition*

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1. Definition

Let M be a differential manifold, TM its tangent bundle, and $\xi \subset TM$ a field of hyperplanes on M, that is, a smooth sub-bundle of codimension 1. Here the terms 'differential' and 'smooth' are used synonymously with C^∞. Locally, ξ can be written as the kernel of a non-vanishing differential 1-form α. A 1-form α defined globally on M with $\xi = \ker \alpha$ can be found if and only if ξ is coorientable, which is equivalent to saying that the quotient line bundle TM/ξ is trivial. The 1-form α is determined by ξ up to multiplication by a smooth function $f\colon M \to \mathbb{R}^*$ or, if the coorientation of ξ has been fixed, by a function taking positive real values only. An equation of the form $\alpha = 0$, with α a non-vanishing 1-form, is classically referred to as a Pfaffian equation.

Definition 1.1. Let M be a smooth manifold of odd dimension $2n+1$. A *contact structure* on M is a hyperplane field $\xi \subset TM$ whose (locally) defining 1-form α has the property that the $(2n+1)$-form $\alpha \wedge (d\alpha)^n$ is nowhere zero, i.e. a volume form, on its domain of definition.

Observe that the condition $\alpha \wedge (d\alpha)^n \neq 0$ is indeed a property of ξ and independent of the choice of defining 1-form α, since

$$(f\alpha) \wedge d(f\alpha)^n = f^{n+1} \alpha \wedge (d\alpha)^n.$$

Definition 1.2. A pair (M, ξ) consisting of an odd-dimensional manifold M and a contact structure ξ on M is called a *contact manifold*.

Definition 1.3. A 1-form as in Definition 1.1, defined globally on M, is called a *contact form* on M.

Occasionally the terminology *strict contact manifold* is used to denote a pair (M, α) consisting of an odd-dimensional manifold and a contact form on it.

2. Examples

2.1. The standard contact structure on $\mathbb{R}^{2n+1}$.

On $\mathbb{R}^{2n+1}$ with Cartesian coordinates

$$(x_1, y_1, \ldots, x_n, y_n, z),$$

the 1-form

$$\alpha_1 := dz + \sum_{j=1}^{n} x_j \, dy_j$$

is a contact form. The contact structure $\xi_1 = \ker \alpha_1$ is called the *standard contact structure* on $\mathbb{R}^{2n+1}$. See Figure 1 for the 3-dimensional case.

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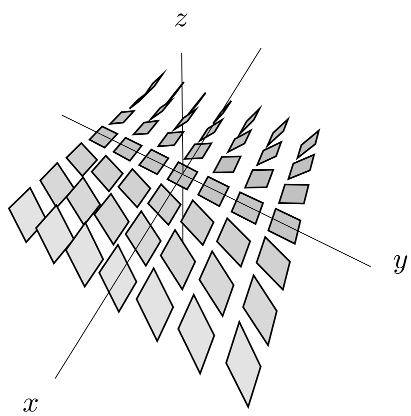

FIGURE 1. The contact structure $\ker(dz + x\,dy)$.

The theorem of Darboux states that locally any contact structure looks like the standard one, cf. [4, Theorem 2.5.1].

Theorem 2.1 (Darboux)**.** *Let α be a contact form on the $(2n+1)$-dimensional manifold M and p a point in M. Then there are coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ on a neighbourhood $U \subset M$ of p such that $p = (0, \ldots, 0)$ and*

$$\alpha|_U = dz + \sum_{j=1}^{n} x_j\, dy_j.$$

A more symmetric form of the standard contact structure on $\mathbb{R}^{2n+1}$ is given by the contact form

$$\alpha_2 := dz + \sum_{j=1}^{n} (x_j\, dy_j - y_j\, dx_j).$$

The contact structure $\xi_2 := \ker \alpha_2$ is equivalent to ξ_1 in the following sense.

Definition 2.2. Two contact manifolds (M, ξ), (M', ξ') are said to be *contactomorphic* if there is a diffeomorphism $\phi \colon M \to M'$ with $T\phi(\xi) = \xi'$, where $T\phi \colon TM \to TM'$ denotes the differential of ϕ. If α, α' are contact forms defining the contact structures ξ, ξ', respectively, this is equivalent to saying that α and $\phi^*\alpha'$ determine the same hyperplane field, and hence equivalent to the existence of a nowhere zero function $f \colon M \to \mathbb{R}^*$ such that $\phi^*\alpha' = f\alpha$.

Usually one deals with contact structures that are cooriented, in which case a contactomorphism is understood to preserve the coorientation. In our examples, one can even find a contactomorphism ϕ where the function f is constant equal to 1. This is called a *strict contactomorphism* of the corresponding strict contact manifolds. In the example, the strict contactomorphism $\phi \colon (\mathbb{R}^{2n+1}, \alpha_1) \to (\mathbb{R}^{2n+1}, \alpha_2)$ is given by

$$\phi(\mathbf{x}, \mathbf{y}, z) = \big((\mathbf{x} + \mathbf{y})/2, (\mathbf{y} - \mathbf{x})/2, z + \mathbf{x}\mathbf{y}/2 \big),$$

where $\mathbf{x}$ and $\mathbf{y}$ stand for $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$, respectively, and $\mathbf{xy}$ stands for $\sum_j x_j y_j$.

2.2. **The standard contact structure on S^{2n+1}.** Let $(x_1, y_1, \dots, x_{n+1}, y_{n+1})$ be Cartesian coordinates on $\mathbb{R}^{2n+2}$. Then the *standard contact structure* ξ_0 *on the unit sphere* S^{2n+1} in $\mathbb{R}^{2n+2}$ is given by the contact form

$$\alpha_0 := \sum_{j=1}^{n+1} (x_j \, dy_j - y_j \, dx_j).$$

Write r for the radial coordinate on $\mathbb{R}^{2n+2}$, that is, $r^2 = \sum_j (x_j^2 + y_j^2)$. One checks easily that $r \, dr \wedge \alpha_0 \wedge (d\alpha_0)^n \neq 0$ for $r \neq 0$. Since S^{2n+1} is a level set of r (or r^2), this verifies the contact condition. Alternatively, one may regard S^{2n+1} as the unit sphere in $\mathbb{C}^{n+1}$. Then the contact structure ξ_0 may be viewed as the hyperplane field of complex tangencies. Indeed, write J for the complex structure on $\mathbb{C}^{n+1}$ corresponding to the complex coordinates $z_j = x_j + \mathrm{i} y_j$, that is, $J(\partial_{x_j}) = \partial_{y_j}$. Then

$$\xi_0 = TS^{2n+1} \cap J(TS^{2n+1}), \tag{1}$$

which means that ξ_0 defines at each point $p \in S^{2n+1}$ the J-invariant subspace of $T_p S^{2n+1}$. Equation (1) follows from the observation that $\alpha_0 = -r \, dr \circ J$.

Here is a further example of contactomorphic manifolds.

Proposition 2.3. *For any point $p \in S^{2n+1}$, the contact manifolds $(S^{2n+1} \setminus \{p\}, \xi_0)$ and $(\mathbb{R}^{2n+1}, \xi_2)$ are contactomorphic.*

This is slightly less obvious than it may seem, since stereographic projection does not quite do the job. For a proof of this proposition, due to Erlandsson, see [4, Proposition 2.1.8].

2.3. **The space of contact elements.** Let B be a smooth n-dimensional manifold. A *contact element* is a hyperplane in a tangent space to B. The *space of contact elements* of B is the collection of pairs (b, V) consisting of a point $b \in B$ and a contact element $V \subset T_b B$. This space of contact elements can be naturally identified with the projectivised cotangent bundle $\mathbb{P}T^*B$, by associating with a hyperplane $V \subset T_b B$ the linear map $u_V \colon T_b B \to \mathbb{R}$, well defined up to multiplication by a non-zero scalar, with $\ker u_V = V$. The space $\mathbb{P}T^*B$ is a manifold of dimension $2n-1$, and it carries a natural contact structure as defined in the following proposition.

Proposition 2.4. *Write π for the bundle projection $\mathbb{P}T^*B \to B$. For $u = u_V \in \mathbb{P}T_b^*B$, let ξ_u be the hyperplane in $T_u(\mathbb{P}T^*B)$ such that $T\pi(\xi_u)$ is the hyperplane V in $T_{\pi(u)}B = T_b B$ defined by u. Then ξ defines a contact structure on $\mathbb{P}T^*B$.*

Figure 2 illustrates the construction for $B = \mathbb{R}^2$. Here $\mathbb{P}T^*B = \mathbb{R}^2 \times \mathbb{R}\mathrm{P}^1$.

Proof of Proposition 2.4. Let $q_1, \dots, q_n$ be local coordinates on B, and $p_1, \dots, p_n$ the corresponding dual coordinates in the fibres of the cotangent bundle T^*B. This means that the coordinate description of covectors is given by

$$(q_1, \dots, q_n, p_1, \dots, p_n) = \left(\sum_{j=1}^{n} p_j \, dq_j \right)_{(q_1, \dots, q_n)}.$$

Thus, a point

$$(q_1, \dots, q_n, (p_1 : \dots : p_n))$$

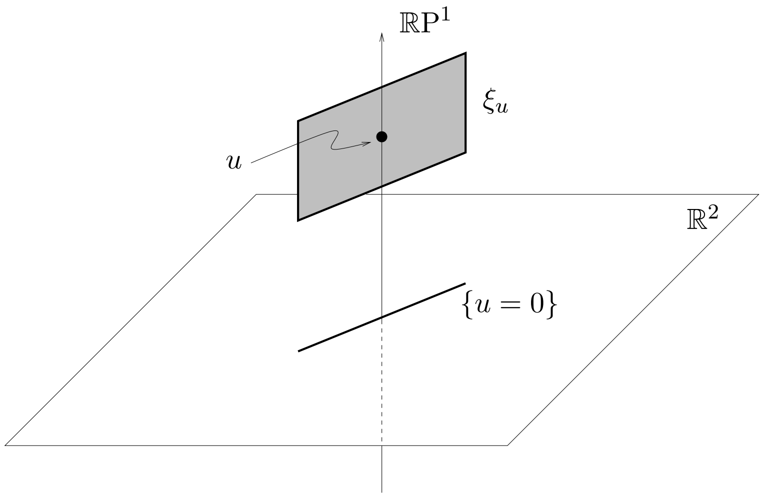

FIGURE 2. The space of contact elements.

in the projectivised cotangent bundle $\mathbb{P}T^*B$ defines the hyperplane

$$\sum_{j=1}^n p_j\, dq_j = 0$$

in T_bB, where $b = (q_1, \dots, q_n)$. By construction, the natural contact structure ξ on $\mathbb{P}T^*B$ is defined by

$$\xi = \ker\Big(\sum_{j=1}^n p_j\, dq_j\Big); \tag{2}$$

notice that this kernel is indeed well defined in terms of the coordinates on $\mathbb{P}T^*B$, although the 1-form $\sum p_j\, dq_j$ is not. In order to verify the contact condition for ξ, we restrict to affine subspaces of the fibre. Over the open set $\{p_1 \neq 0\}$, for instance, ξ is defined in terms of affine coordinates $p_j' = p_j/p_1$, $j = 2, \dots, n$, by the equation

$$dq_1 + p_2'\, dq_2 + \cdots + p_n'\, dq_n = 0,$$

which is exactly the description of the standard contact structure on $\mathbb{R}^{2n-1}$. $\square$

2.4. **A non-coorientable contact structure.** In the previous example, we now specialise to $B = \mathbb{R}^n$. Then the space of contact elements is $\mathbb{P}T^*B = \mathbb{R}^n \times \mathbb{R}\mathrm{P}^{n-1}$. In terms of Cartesian coordinates $(q_1, \dots, q_n)$ on $\mathbb{R}^n$ and homogeneous coordinates $(p_1 : \dots : p_n)$ on $\mathbb{R}\mathrm{P}^{n-1}$, the natural contact structure on this space of contact elements is now defined globally by equation (2). For $n = 2$, and identifying $\mathbb{R}\mathrm{P}^1$ with $\mathbb{R}/\pi\mathbb{Z}$ with coordinate θ, this natural contact structure can be written as

$$\ker(\sin\theta\, dx - \cos\theta\, dy). \tag{3}$$

This is an example of a contact structure that is not coorientable. It lifts to a coorientable contact structure on $\mathbb{R}^2 \times S^1$, given by the same equation, with $S^1 := \mathbb{R}/2\pi\mathbb{Z}$. Similar orientability issues arise for general n. Write $M := \mathbb{P}T^*\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}\mathrm{P}^{n-1}$ and ξ for the natural contact structure on this space of contact elements. We claim the following:

(i) If n is even, then M is orientable; ξ is neither orientable nor coorientable.

(ii) If n is odd, then M is not orientable; ξ is not coorientable, but it is orientable.

The statement about orientability of M follows from the corresponding statement for $\mathbb{R}\mathrm{P}^{n-1}$. The fact that ξ is never coorientable follows from the observation that TM/ξ can be identified with the canonical line bundle on $\mathbb{R}\mathrm{P}^{n-1}$ (pulled back to M), which is known to be non-trivial, see [4, Proposition 2.1.13]. In case (i), since M is orientable but ξ not coorientable, it follows that ξ cannot be orientable. The fact that in case (ii) the contact structure is orientable is the consequence of a more general statement in the next section.

2.5. **More orientability issues.** Notice that a contact manifold with a coorientable contact structure is always orientable (and so is the contact structure), because a globally defined contact form gives rise to a volume form on the manifold. This gives a quicker way to see that in our previous example for n odd the contact structure ξ cannot be coorientable. But even for contact structures that need not be coorientable one has the following:

(i) Any contact manifold of dimension $4n-1$ is naturally oriented.
(i) Any contact structure on a manifold of dimension $4n+1$ is naturally oriented.

Statement (i) follows from the observation that the sign of the volume form $\alpha \wedge (d\alpha)^{2n-1}$ does not depend on the choice of (local) 1-form α defining the contact structure. Similarly, in case (ii) the sign of $(d\alpha)^{2n}|_\xi$ does not depend on the choice of α.

2.6. **Three-dimensional contact manifolds.** One can easily write down examples of contact structures on some closed 3-manifolds. The 3-sphere is dealt with in Section 2.2. The contact structure from equation (3) in Section 2.4 descends to a contact structure on the 3-torus $T^3 = \mathbb{R}^3/(2\pi\mathbb{Z})^3$. On $S^1 \times S^2 \subset S^1 \times \mathbb{R}^3$ one has the contact structure $\ker(z\, d\theta + x\, dy - y\, dx)$. Notice that by the previous section a 3-dimensional contact manifold is necessarily orientable. In fact, as shown by Martinet [9], this is the only restriction.

Theorem 2.5 (Martinet)**.** *Every closed, orientable 3-manifold admits a contact structure.*

2.7. **Brieskorn manifolds.** Let $\mathbf{a} = (a_0, \dots, a_n)$ be an $(n+1)$-tupel of integers $a_j > 1$, and set

$$V(\mathbf{a}) := \{\mathbf{z} := (z_0, \dots, z_n) \in \mathbb{C}^{n+1} : f(\mathbf{z}) := z_0^{a_0} + \cdots + z_n^{a_n} = 0\}.$$

Further, with S^{2n+1} denoting the unit sphere in $\mathbb{C}^{n+1}$, we define

$$\Sigma(\mathbf{a}) := V(\mathbf{a}) \cap S^{2n+1}.$$

This turns out to be a smooth manifold of dimension $2n-1$. Manifolds of this form are called *Brieskorn manifolds* [2]. It can be shown that the standard contact form on S^{2n+1} induces a contact structure on $\Sigma(\mathbf{a})$. This has been observed independently by Abe–Erbacher, Lutz–Meckert and Sasaki–Hsü, cf. [4, Section 7.1].

3. A brief history of the terminology

The concept of a contact element first appeared in systematic form in 1896 in the work of Sophus Lie [7]. His terminology was a little more specific, for instance, a contact element of the plane was called a line element (Linienelement). A contact transformation (Berührungstransformation) for Lie was defined as above, but he only considered this in the context of spaces of contact elements and their natural contact structure (which did not yet bear that name). Such contact transformations play a significant role in the work of E. Cartan, E. Goursat, H. Poincaré and others in the second half of the 19th century. For instance, the Legendre transformation in classical mechanics is a contact transformation. The study of contact manifolds in the modern sense can be traced back to the work of Georges Reeb [10], who referred to a strict contact manifold (M, α) as a 'système dynamique avec invariant intégral de Monsieur Elie Cartan'. The relation with dynamical systems comes from the fact that a contact form α gives rise to a vector field R defined uniquely by the equations

$$d\alpha(R, .\,) \equiv 0 \quad \text{and} \quad \alpha(R) \equiv 1.$$

This vector field is nowadays called the *Reeb vector field* of α. The words 'contact structure' and 'contact manifold' seem to make their first appearance in the work of Boothby-Wang [1], Gray [5] and Kobayashi [6] in the late 1950s. For more historical information on contact manifolds see [8] and [3].

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