Bulletin of the Manifold Atlas - definition (2013)

Totally geodesic submanifold - definition*

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1. Definition

We consider a submanifold M of a Riemannian manifold $(\overline{M}, \overline{g})$. The Riemannian metric $\overline{g}$ induces a Riemannian metric g on the submanifold M. Then (M, g) is also called a *Riemannian submanifold* of the Riemannian manifold $(\overline{M}, \overline{g})$.

Definition 1.1. A submanifold M of a Riemannian manifold $(\overline{M}, \overline{g})$ is called *totally geodesic* if any geodesic on the submanifold M with its induced Riemannian metric g is also a geodesic on the Riemannian manifold $(\overline{M}, \overline{g})$.

General references are [1, ch.11] and [2, I §14]. On the Riemannian manifold (M, g) resp. $(\overline{M}, \overline{g})$ there exists an unique torsion free and metric connection $\overline{\nabla}$ resp. ∇. It is called the Levi-Civita connection. Then the *shape tensor* or *second fundamental form tensor* II is a symmetric tensor field which can be defined as follows for tangent vectors $X, Y \in T_pM$ resp. vector fields X, Y on the submanifold:

$$II(X, Y) = \overline{\nabla}_X Y - \nabla_X Y\,. \tag{1}$$

Proposition 1.2 (cf. [4, p.104])**.** *For a Riemannian submanifold (M, g) of the Riemannian manifold $(\overline{M}, \overline{g})$ the following statements are equivalent:*

(a) *(M, g) is a totally geodesic submanifold of $(\overline{M}, \overline{g})$.*
(b) *The shape tensor vanishes: $II = 0$.*
(c) *For a vector v tangential to the submanifold M the geodesic γ on the Riemannian manifold $(\overline{M}, \overline{g})$ defined on a small interval $(-\epsilon, \epsilon)$ with initial direction $\gamma'(0) = v$ stays on the submanifold.*

Part (c) implies that locally a totally geodesic submanifold $M \subset \overline{M}$ is uniquely determined by the vector subspace $T_pM \subset T_p\overline{M}$ for some $p \in M$, provided that M is connected and complete. There is a result by É. Cartan providing necessary and sufficient conditions for the existence of a totally geodesic submanifold tangential to a given vector subspace V of the tangent space $T_p\overline{M}$ in terms of the curvature tensor, cf. [1, 11.1]. This result shows that for most Riemannian manifolds no totally geodesic submanifolds of dimension at least two exist. On the other hand totally geodesic submanifolds do occur if the manifold carries isometries:

Theorem 1.3 (cf. [3, 1.10.15])**.** *Let $f : (\overline{M}, \overline{g}) \longrightarrow (\overline{M}, \overline{g})$ be an isometry of the Riemannian manifold $(\overline{M}, \overline{g})$. Then every connected component M of the fixed point set*

$$\left\{ y \in \overline{M}; f(y) = y \right\} \tag{2}$$

with the induced Riemannian metric is a totally geodesic submanifold.

*Atlas page: www.map.mpim-bonn.mpg.de/Totally_geodesic_submanifold

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2. EXAMPLES

Example 2.1.

(a) A geodesic $\gamma : \mathbb{R} \to M$ can be viewed as a totally geodesic submanifold of dimension one.

(b) Consider the standard sphere

$$S^n := \left\{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \,;\, x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\right\}.$$

For $1 \leq k < n$ the k-sphere

$$S^k = \{(x_1, x_2, \ldots, x_{n+1}) \in S^n \,;\, x_{k+1} = \ldots = x_{n+1} = 0\}$$

is a totally geodesic submanifold of S^n. It is the fixed point set of the isometry $f : S^n \to S^n$:

$$f(x_1, x_2, \ldots, x_{n+1}) = (x_1, x_2, \ldots, x_k, -x_{k+1}, \ldots, -x_{n+1}).$$

One can see immediately that any complete k-dimensional totally geodesic submanifold of S^n is of this form up to an isometry of the sphere.

(c) We denote by $P^n(\mathbb{C})$ the *n-dimensional complex projective space* of one-dimensional linear subspaces of the complex vector space $\mathbb{C}^{n+1}$. For $1 \leq k < n$ the inclusion $(z_1, \ldots, z_{k+1}) \in \mathbb{C}^k \mapsto (z_1, \ldots, z_{k+1}, 0, \ldots, 0) \in \mathbb{C}^n$ induces an inclusion of the k-dimensional complex projective space $P^k(\mathbb{C})$ into $P^n(\mathbb{C})$. This is a totally geodesic submanifold since it is the fixed point set of the isometry on $P^n(\mathbb{C})$ induced by the reflection $(z_1, z_2, \ldots, z_{n+1}) \mapsto (z_1, z_2, \ldots, z_{k+1}, -z_{k+2}, \ldots, -z_{n+1})$.

The last two examples are in particular examples of *symmetric spaces*. The totally geodesic submanifolds of a symmetric space can be described in terms of a Lie triple system, cf. [2, ch.IV, §7] or [1, 11.2].

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