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1-manifolds*

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Abstract. This is hopefully a complete account of the topology of 1-dimensional manifolds. In general, this material is not well represented in the literature. This probably happened because it is considered too simple. Hence it is difficult to find appropriate references. The proofs are very simple, but use ideas that are less applicable in similar high-dimensional situations. This is why most the results below are provided with proofs.

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1. Introduction

According to the general definition of manifold, a manifold of dimension 1 is a topological space which is second countable (i.e., its topological structure has a countable base), satisfies the Hausdorff axiom (any two different points have disjoint neighborhoods) and each point of which has a neighbourhood homeomorphic either to the real line $\mathbb{R}$ or to the half-line $\mathbb{R}_+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

Manifolds of dimension 1 are called *curves*, but this name may lead to a confusion, because many mathematical objects share it. Even in the context of topology, the term *curve* may mean not only a manifold of dimension 1 with an additional structure, but, for instance, an immersion of a smooth manifold of dimension 1 to Euclidean space $\mathbb{R}^n$. To be on the safe side, we use an unambiguous term *manifold of dimension* 1 or 1-*manifold*.

The material presented below is not well represented in the literature. This happened, probably, because it is considered too simple. It was difficult to find appropriate references. The proofs are really elementary, but use ideas that are barely applicable in similar high-dimensional situations. That's why most the results below are provided with proofs.

Besides the adjacent field of dynamics, there is no research activity related to the topology of 1-manifolds. The dynamical topics have been left outside the scope of this text. In particular, no infinite group actions are considered, although actions of finite groups on 1-manifolds are on this side of the natural borderline, and a complete account of them is presented.

For other expositions about 1-manifolds, see [3], [2] and also [1, Sections 3.1.1.16-19].

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2. Construction and examples

2.1. Examples of connected 1-manifolds.

- The real line: $\mathbb{R}$
- The half-line: $\mathbb{R}_+$
- The circle: $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$
- The closed interval: $I = [0, 1]$

2.2. Doubling. If X is an n-manifold and ∂X is its *boundary* (i.e., the set of points of X that do not have neighborhoods homeomorphic to the Euclidean space $\mathbb{R}^n$), then the quotient of the disjoint union $X \amalg X$ of two copies of X by the identity map $\partial X \to \partial X$ is an n-manifold with empty boundary, called the *double of* X and denoted by $\mathrm{D}(X)$.

This operation is well defined up to homeomorphism. It gives a natural embedding of a manifold with boundary into a manifold without boundary (i.e., with empty boundary) and allows one to reduce many problems about manifolds with boundary to problems about manifolds without boundary.

Examples:

- $\mathrm{D}(\mathbb{R}_+) = \mathbb{R}$
- $\mathrm{D}(I) = S^1$

3. Topological classification

3.1. Reduction to classification of connected manifolds. The following elementary facts hold for n-manifolds of any dimension n:

Any manifold is homeomorphic to the disjoint sum of its connected components.

A connected component of an n-manifold is a n-manifold.

Two manifolds are homeomorphic iff there exists a one-to-one correspondence between their components such that the corresponding components are homeomorphic.

3.2. Topological classification of connected 1-manifolds.

Theorem 3.1. *Any connected* 1*-manifold is homeomorphic to one of the following four manifolds:*

(1) *real line* $\mathbb{R}$*,*
(2) *half-line* $\mathbb{R}_+$*,*
(3) *circle* $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$*,*
(4) *closed interval* $I = [0, 1]$*.*

No two of these manifolds are homeomorphic to each other.

3.3. Characterizing the topological type of a connected 1-manifold.

Theorem 3.2. (1) *Any connected non-compact* 1*-manifold without boundary is homeomorphic to* $\mathbb{R}$*.*
(2) *Any connected non-compact* 1*-manifold with non-empty boundary is homeomorphic to* $\mathbb{R}_+$*.*
(3) *Any connected closed* 1*-manifold is homeomorphic to* S^1*.*

(4) *Any connected compact* 1*-manifold with non-empty boundary is homeomorphic to* I*.*

Thus, for connected 1-manifolds, two invariants, compactness and presence of boundary, form a complete system of topological invariants. Each of the invariants takes two values.

Theorems 3.1 and 3.2 above solve the topological classification problem for 1-manifolds in the most effective way that one can desire. Surprisingly, many Topology textbooks manage not to mention this fundamental result.

3.4. **About proofs of the classification theorems.** The proofs of Theorems 3.1 and 3.2 above are elementary. They can be found, e.g., in [1, Sections 3.1.1.16-19] and [2]. In the case of 1-manifolds without boundary, the proofs are based on the following simple lemmas:

Lemma 3.3. *Any connected* 1*-manifold covered by two open sets homeomorphic to* $\mathbb{R}$ *is homeomorphic either to* $\mathbb{R}$ *or to* S^1*.*

Under assumptions of Lemma 3.3, the 1-manifold is homeomorphic to $\mathbb{R}$ iff the intersection of two open sets is connected, and it is homeomorphic to S^1 iff the intersection consists of two connected components (the intersection cannot have more than two components).

Lemma 3.4. *If a topological space* X *can be represented as the union of a nondecreasing sequence of open subsets, all homeomorphic to* $\mathbb{R}$*, then* X *is homeomorphic to* $\mathbb{R}$*.*

From the topological classification for 1-manifolds without boundary, the classification for 1-manifolds with non-empty boundary is obtained using the doubling operation, see Section 2.2 above.

3.5. **Corollary: homotopy classification.**

Theorem 3.5. *Each connected* 1*-manifold is either contractible, or homotopy equivalent to circle.*

This follows immediately from Theorem 3.1.

3.6. **Corollary: Cobordant 0-manifolds.**

Theorem 3.6. *A compact* 0*-manifold* X *bounds a compact* 1*-manifold iff the number of points in* X *is even.*

Corollary 3.7. *Two compact* 0*-manifolds are cobordant iff their numbers of points are congruent modulo* 2*.*

3.7. **Characterizing connected 1-manifolds in terms of separating points.** A subset A of a topological space X is said to *separate* X if $X \smallsetminus A$ can be presented as a union of two disjoint open sets.

Theorem 3.8. *(See* [4]*.) Let* X *be a connected compact Hausdorff second countable topological space.*

(1) *If every two points separate* X*, then* X *is homeomorphic to the circle.*

(2) *If each point, with two exceptions, separates X, then X is homeomorphic to I.*

Any point $a \in \mathbb{R}$ splits $\mathbb{R}$ to two disjoint open rays $(-\infty, a) = \{x \in \mathbb{R} \mid x < a\}$ and $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$.

Theorem 3.9. *(See [5].) Let X be a connected locally compact Hausdorff second countable topological space.*

(1) *If the complement of each point in X consists of two connected components, then X is homeomorphic to $\mathbb{R}$.*
(2) *If X contains a point b such that $X \smallsetminus b$ is connected and $X \smallsetminus a$ consists of two connected components for each $a \in X$, $a \neq b$, then X is homeomorphic to $\mathbb{R}_+$.*

4. Orders and orientations

4.1. **Interval topology.** Most properties specific for 1-manifolds can be related to the fact that the topological structure on a connected 1-manifold is defined by linear or cyclic ordering of its points.

Open intervals $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ form a base of the standard topology on $\mathbb{R}$. This way of introducing a topological structure can be applied in any (linearly) ordered set X (though in a general linearly ordered set one should include into the base, together with open intervals $(a, b) = \{x \in X \mid a < x < b\}$, also open rays $\{x \in X \mid x < a\}$ and $\{x \in X \mid a < x\}$). On $\mathbb{R}_+$ and I, the standard topology is induced from the standard topology on $\mathbb{R}$, and can be described in terms of the order.

Theorem 4.1. *Every connected non-closed 1-manifold admits exactly two linear orders defining its topology.*

Proof. A linear order $\prec$ on a set X is encoded in the system of rays $\{x \in X \mid a \prec x\}$ for $a \in X$.

By Theorem 3.2, a connected non-closed 1-manifold is homeomorphic either to $\mathbb{R}$, or $\mathbb{R}_+$, or I. On each of these 1-manifolds there are two linear orders, $<$ and $>$, defining the topology. For these orders, the rays $U = \{x \in X \mid x < a\}$ and $V = \{x \in X \mid a < x\}$ are defined by the topology: they are just the connected components of $X \smallsetminus a$.

For any other linear order $\prec$ defining the same topology on X, the rays $\{x \in X \mid x \prec a\}$ and $\{x \in X \mid a \prec x\}$ are open and intersect the connected components U and V of $X \smallsetminus a$ in disjoint open sets. By connectedness of U and V, one of them coincides with U, the other with V. Hence, $\prec$ coincides with one of the standard orders, either with $<$, or $>$. □

4.2. **Orientations.** An orientation of a 1-manifold can be interpreted via linear orderings on its open subsets homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$. An orientation of $\mathbb{R}$ or $\mathbb{R}_+$ is nothing but one of the two linear orders defining the topological structure.

Relation to the general homological definition of orientation. Recall that in high dimensional situations orientation of an open set $U \subset \mathbb{R}^n$ is defined as a coherent choice of generators in homology groups $H_n(U, U \smallsetminus x)$ for $x \in U$. In our case $n = 1$ and the group $H_n(U, U \smallsetminus x) = H_1(\mathbb{R}, \mathbb{R} \smallsetminus x)$ is generated by a homology class of a singular cycle consisting of a single singular 1-simplex $f : I \to \mathbb{R}$, which is an embedding with $x \in f(\operatorname{int} I)$. There are two generators: one is represented by a monotone increasing f, another, by a monotone decreasing f. A choice of linear order on $\mathbb{R}$ allows one to distinguish one of the generators of $H_1(\mathbb{R}, \mathbb{R} \smallsetminus x)$: namely, the one represented by f which is monotone increasing with respect to the order chosen on $\mathbb{R}$.

For extending the notion of orientation to a general 1-manifold, one needs to globalize the idea of linear order. It can be done in several ways.

For example, due to the topological classification, one can restrict to just four model 1-manifolds: $\mathbb{R}$, $\mathbb{R}_+$, I and S^1. For $\mathbb{R}$, $\mathbb{R}_+$ and I, an orientation still can be defined as a linear order determining the topology of the manifold. For S^1 this approach does not work, but can be adjusted: instead of linear order one can rely on cyclic orders that define the topology. However, this is a bit awkward, as cyclic orders are more cumbersome than usual linear orders.

There is a more conceptual approach, which imitates the classical definition of orientations of differentiable manifolds, but relies, instead of coordinate charts, on local linear orders.

Let X be a 1-manifold. A *local order* of X is a pair consisting of an open set $U \subset X$ homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$ and a linear order on U defining the topology on U. Two local orders $(U, <_U)$, $(V, <_V)$ are said to *agree* if on any connected component W of $U \cap V$ the orders $<_U$ and $<_V$ induce the same order.

Denote by $\operatorname{LocOrd}(X)$ the set of all local orders of X. An *orientation* on X is a map $o : \operatorname{LocOrd}(X) \to \{+1, -1\}$ such that for any $(U, <_U), (V, <_V) \in \operatorname{LocOrd}(X)$ and the restrictions of $<_U$ and $<_V$ to any connected component of $U \cap V$ coincide if $o(U, <_U) = o(V, <_V)$ and do not coincide if $o(U, <_U) \neq o(V, <_V)$.

Obvous Lemma 4.2. *Let $\mathcal{U}$ be a collection of open sets in a 1-manifold X homeomorphic to $\mathbb{R}$ and let for any open set $V \subset X$ homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$ there exist $U \in \mathcal{U}$ such that $U \cap V$ is connected. If each $U \in \mathcal{U}$ is equipped with a linear order $<_U$ defining the topology on U such that the local orders $(U, <_U)$ and $(V, <_V)$ agree for any $U, V \in \mathcal{U}$, then there exists a unique orientation o on X such that $o(U, <_U) = +1$ for any $U \in \mathcal{U}$. Moreover any orientation on X comes from such coherent linear orders $<_U$ on all elements of $\mathcal{U}$.*

Theorem 4.3. *On any connected 1-manifold there exists exactly two orientations.*

Proof. If X is a non-closed connected 1-manifold, then for $\mathcal{U}$ satisfying the hypothesis of Lemma 4.2 we can take a collection consisting of a single element $\operatorname{int} X$. If X is closed connected 1-manifold, then for $\mathcal{U}$ we can take the collection of complements of single points. Then the intersection $U \cap V$ for any $U, V \in \mathcal{U}$ consists of two connected components homeomorphic to $\mathbb{R}$. We can choose stereographic projections as homeomorphisms of them to $\mathbb{R}$ in such a way that the transition mapping from one of these charts to another one is $x \mapsto a - \frac{1}{x-b}$. It is monotone increasing on each of the rays $(-\infty, b)$ and $(b, +\infty)$. Therefore the orders obtained on complements of points via these stereographic projections from the standard order $<$ on $\mathbb{R}$ agree with each other and we can apply Lemma 4.2. $\square$

Corollary 4.4. *Any* 1*-manifold admits an orientation. If a* 1*-manifold consists of* n *connected components, then it admits exactly* 2^n *orientations.*

4.3. **Self-homeomorphisms.**

Theorem 4.5. *A map* $h : \mathbb{R} \to \mathbb{R}$ *is a homeomorphism iff* h *is a monotone bijection.*

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be a homeomorphism. First, observe that h maps every ray to a ray. Indeed, for any $x \in \mathbb{R}$, the map h induces a homeomorphism $\mathbb{R} \smallsetminus x \to \mathbb{R} \smallsetminus h(x)$. The rays $(-\infty, x)$ and (x, ∞) are connected components of $\mathbb{R} \smallsetminus x$. Therefore their images are connected components $(-\infty, h(x))$ and $(h(x), \infty)$ of $\mathbb{R} \smallsetminus h(x)$.

Observe that rays have the same direction iff one of them is contained in the other one. Therefore two rays of the same direction are mapped by h to rays with the same direction. Thus rays $(x, +\infty)$ are mapped either all to rays $(h(x), +\infty)$ or all to $(-\infty, h(x))$. Thus h is monotone.

Let $h : \mathbb{R} \to \mathbb{R}$ be a monotone bijection. Then the image and preimage under h of any open interval are open intervals. Therefore, both h and h^{-1} are continuous, and hence h is a homeomorphism. □

The following theorem can be proved similarly or can be deduced from Theorem 4.5.

Theorem 4.6.

(1) *A map* $h : I \to I$ *is a homeomorphism iff* h *is a monotone bijection.*
(2) *A map* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *is a homeomorphism iff* h *is a monotone increasing bijection.*
(3) *A map* $h : S^1 \to S^1$ *is a homeomorphism iff* h *is a bijection that either preserves or reverses the cyclic order of points on* S^1*.*

A self-homeomorphism $h : X \to X$ of a connected 1-manifold maps an orientation o of X to some orientation, which is either o or $-o$ (as a connected 1-manifold, X has just these two orientations). Whether h maps o to itself or to the opposite orientation $-o$ does not depend on o: it maps o to o iff it maps $-o$ to $-o$. We say that h is *orientation preserving* if it maps any orientation of X to itself and *orientation reversing* if if it maps any orientation of X to the opposite orientation.

The half-line $\mathbb{R}_+$ does not admit a self-homeomorphism reversing orientation. Any connected 1-manifold non-homeomorphic to $\mathbb{R}_+$ admits an orientation reversing map. Thus, $\mathbb{R}_+$ is chiral and connected 1-manifolds non-homeomorphic to $\mathbb{R}_+$ are amphicheiral.

Thus, there are 5 topological types of oriented connected 1-manifolds: the topological type of the non-oriented half-line splits into the oriented topological types of $\mathbb{R}_+$ and $\mathbb{R}_-$ with the orientations induced by the standard order.

5. INVARIANTS

5.1. **Basic invariants.** As follows from the Theorems 3.1 and 3.2 above, the following invariants:

- the number of connected components,
- the compactness of each connected component,

• and the number of boundary points of each connected component,

determine the topological type of a 1-manifold.

5.2. **Homology groups.** The low dimensional homology groups of 1-manifolds are presented in the following table:

Homology group \ 1-manifold	$\mathbb{R}$	$\mathbb{R}_+$	S^1	I	$a\mathbb{R} \amalg b\mathbb{R}_+ \amalg cS^1 \amalg dI$
$H_0(X)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^{a+b+c+d}$
$H_1(X)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}^c$
$H_0(X, \partial X)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}^{a+c}$
$H_1(X, \partial X)$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^{c+d}$

It is clear from the last column of this table that the absolute and relative homology groups of dimensions 1 and 2 determine the topological type of 1-manifold. Of course, the basic invariants considered above do the job in a more elementary way.

Above by homology we mean homology with compact support. For homology with closed support (Borel-Moore homology) see the following table:

Borel-Moore homology group \ 1-manifold	$\mathbb{R}$	$\mathbb{R}_+$	S^1	I	$a\mathbb{R} \amalg b\mathbb{R}_+ \amalg cS^1 \amalg dI$
$H_0^{BM}(X)$	0	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^{c+d}$
$H_1^{BM}(X)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}^{a+c}$
$H_0^{BM}(X, \partial X)$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}^c$
$H_1^{BM}(X, \partial X)$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}^{a+b+c+d}$

An orientation of a 1-manifold X gives rise to the fundamental class of X, which belongs to $H_1^{BM}(X, \partial X)$. Cap-product by this class defines various versions of the Poincaré duality isomorphisms between usual cohomology (recall that the usual cohomology has closed support) and the relative Borel-Moore homology of the complementary dimension. So there are isomorphisms

$$H^1(X; \mathbb{Z}) \to H_0^{BM}(X, \partial X), \quad H^0(X; \mathbb{Z}) \to H_1^{BM}(X, \partial X),$$
$$H^1(X, \partial X; \mathbb{Z}) \to H_0^{BM}(X), \quad H^0(X, \partial X; \mathbb{Z}) \to H_1^{BM}(X).$$

A local coefficient system on a 1-manifold homeomorphic to the circle, may be non-trivial. E.g., if the local coefficient system over S^1 with fibre $\mathbb{C}$ has non-trivial monodromy, then all the homology groups are trivial.

5.3. **Euler characteristics.** The Euler characteristic of I is 1. The Euler characteristic of S^1 is 0.

Any closed 1-manifold has Euler characteristic 0. The Euler characteristic of any compact 1-manifold is half the number of its boundary points.

When it comes to non-compact spaces, the notion of Euler characteristic becomes ambiguous. A non-compact space has many Euler characteristics. The notion of Euler characteristic depends on the properties that one wants to preserve. If one wants to have invariance under homotopy equivalences, then the additivity would be lost. If one wants to keep additivity, then it is not the alternating sum of Betti numbers. If one wants to keep topological invariance and additivity, then $\chi(\mathbb{R}) = \chi(I) - 2\chi(\text{pt}) = 1 - 2 = -1$, similarly $\chi(\mathbb{R}_+) = 0$.

For not necessarily compact 1-manifolds probably most useful is the Euler characteristic which is defined as the alternating sum of the ranks of homology groups with closed support (Borel-Moore homology). It is additive and a topological invariant. The only property which may confuse a person with purely compact experience is that it is not homotopy invariant. With this Euler characteristic, $\chi(\mathbb{R}) = -1$ and $\chi(\mathbb{R}_+) = 0$.

5.4. **Other homotopy invariants.** The homotopy invariants of 1-manifolds are extremely simple. All homology and homotopy groups of dimensions > 1 are trivial. The fundamental group $\pi_1(X, x_0)$ is an infinite cyclic group, if the connected component of X containing x_0 is homeomorphic to circle, and trivial otherwise.

5.5. **Tangent bundle invariants.** The tangent bundles of 1-manifolds are trivial. Thus all the characteristic classes are trivial.

6. Additional structures

6.1. **Triangulations.** A triangulation of a 1-manifold X is a locally finite cover of X by subspaces homeomorphic to I, any two of which have disjoint interiors and at most one common point. The subspaces are assumed to be equipped with affine structure or, rather, with homeomorphisms to I. The subspaces are called *edges* or 1-*simplices*, the images of the endpoints of I are called *vertices* or 0-*simplices*.

Any 1-manifold admits a triangulation. A triangulation of a non-compact connected 1-manifold is unique up to homeomorphism.

A compact 1-manifold has non-homeomorphic triangulations, but they are easy to classify up to homeomorphism. On the circle the topological type of a triangulation is defined by the number of 1-simplices. This number can take any integral value ≥ 3. Similarly, the topological type of a triangulation of I is defined by the number of 1-simplices, which can take any positive integral value.

6.2. **Metrics and intrinsic metrics.** Recall that any manifold is metrizable. On a connected manifold, a metric defining its topology can be replaced by an intrinsic metric defining the same topology. (Recall that a metric on a path-connected space is said to be *intrinsic* if the distance between any two points is equal to the infimum of lengths of paths connecting the points, and that the length of a path $s : I \to X$ in a metric space X with metric $d : X \times X \to \mathbb{R}_+$ is $\sup\{\sum_{i=1}^n d(s(t_{i-1}), s(t_i)) \mid$ all sequences $0 = t_0 < t_1 < \cdots < t_n = 1\}$.)

A connected 1-manifold with an intrinsic metric is defined up to isometry by the diameter of the space. Recall that the diameter of a metric space X with metric $d : X \times X \to \mathbb{R}_+$ is $\sup\{d(x, y) \mid x, y \in X\}$.

For each value of the diameter and each homeomorphism type of a connected 1-manifold with intrinsic metric there is a unique standard model:

Theorem 6.1. *A 1-manifold X with intrinsic metric and diameter D is isometric to*

- $(-D/2, D/2)$ *with the metric induced from the standard metric on* $\mathbb{R}$ *and any* $D \in (0, \infty]$;

- $[0, D)$ *with the metric induced from the standard metric on* $\mathbb{R}$ *and any* $D \in (0, \infty]$;
- *the circle* $\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = D^2/\pi^2\}$ *with the intrinsic metric defined by the lengths of paths on the circle measured in* $\mathbb{R}^2$ *and any* $D \in (0, \infty)$;
- $[0, D]$ *with the metric induced from the standard metric on* $\mathbb{R}$ *and any* $D \in (0, \infty)$.

The isometries to the standard models can be constructed using distances from some points. For example, for a connected 1-manifold X homeomorphic to $\mathbb{R}_+$ equipped with an intrinsic metric, the distance to the only boundary point defines a canonical isometry $X \to [0, D)$.

6.3. Smooth structures.

Theorem 6.2. *Any* 1*-manifold admits a smooth structure of any class* C^r.

Proof. A smooth structure can be induced by the isometry to the corresponding standard model from Theorem 6.1 above. □

If smooth 1-manifolds X and Y are homeomorphic, then they are also diffeomorphic. Moreover,

Theorem 6.3. *Any homeomorphism between two smooth* 1*-manifolds can be approximated in the* C^0*-topology by a diffeomorphism.*

Proof. By Theorems 4.5 and 4.6, a homeomorphism is monotone in the appropriate sense. Choose a net of points in the source such that the image of each of them is sufficiently close to the images of its neighbors. Take a smooth monotone bijection coinciding with the homeomorphism at the chosen points. □

7. Constructions

7.1. Surgery. Any compact 1-manifold can be transformed by surgeries to any other 1-manifold with the same boundary.

If two compact 1-manifolds with the same boundary are oriented and the induced orientations on the boundary coincide, then the surgery can be chosen to preserve the orientation (this means that the corresponding cobordism is an oriented 2-manifold and its orientation induces on the boundary the given orientation on one of the 1-manifolds and the orientation opposite to the given one on the other 1-manifold).

An index-1 surgery preserving orientation on a closed 1-manifold changes the number of connected components by 1. An index-1 surgery on a 1-manifold, which does not preserve every orientation, preserves the number of connected components. In particular, if an index-1 surgery on a connected 1-manifold does not preserve an orientation, then its result is a connected 1-manifold.

7.2. Connected sums. The notion of connected sum is defined for 1-manifolds, but the connectivity of the outcome is different in dimension 1 compared to other dimensions.

Indeed, the term *connected sum* can be misleading in dimension 1 since a connected sum of connected 1-manifolds may be not connected. For example a connected sum of two copies of $\mathbb{R}$ is a disjoint sum of two copies of $\mathbb{R}$.

Note that connected sum is only a well defined operation on *oriented manifolds* and one has to be careful with the orientations. For example

$$\mathbb{R}_+\sharp\mathbb{R}_+ \cong \mathbb{R}_+ \amalg \mathbb{R}_+ \quad \text{but} \quad \mathbb{R}_+\sharp(-\mathbb{R}_+) \cong I \amalg \mathbb{R}.$$

8. Groups of self-homeomorphisms

8.1. **Mapping class groups.** Recall that the *mapping class group* of a manifold X is the quotient group of the group $\mathrm{Homeo}(X)$ of all homeomorphisms $X \to X$ by the normal subgroup of homeomorphisms isotopic to the identity. In other words, the mapping class group of X is $\pi_0(\mathrm{Homeo}(X))$.

An orientation reversing homeomorphism cannot be isotopic to an orientation preserving homeomorphism. For auto-homeomorphisms of a connected 1-manifold this is the only obstruction to being isotopic:

Theorem 8.1. *Any two auto-homeomorphisms of a connected* 1*-manifold that are either both orientation preserving, or both orientation reversing are isotopic.*

This is a corollary of the following two obvious lemmas.

Lemma 8.2. On rectilinear isotopy. *Let X be one of the following* 1*-manifolds:* $\mathbb{R}$*,* $\mathbb{R}_+$*, or I. Let $f, g : X \to X$ be two monotone bijections that are either both increasing or both decreasing. Then the family $h_t = (1-t)f + tg : X \to X$ with* $t \in [0,1]$ *consists of monotone bijections (and hence is an isotopy between f and g).*

Lemma 8.3. *Let $f, g : S^1 \to S^1$ be two bijections that either both preserve or both reverse the standard cyclic order of points on S^1. Let f and g coincide at $x \in S^1$. Then f and g are isotopic via the canonical isotopy which is stationary at x and is provided on the complement of x by stereographic projections and the rectilinear isotopy from Lemma 8.2 of the corresponding self-homeomorphisms of* $\mathbb{R}$*.*

Corollary 8.4. $\pi_0(\mathrm{Homeo}(S^1)) \cong \pi_0(\mathrm{Homeo}(\mathbb{R})) \cong \pi_0(\mathrm{Homeo}(I)) \cong \mathbb{Z}/2$ *and* $\pi_0(\mathrm{Homeo}\,\mathbb{R}_+) \cong 0$.

Remark. All the statements in this section remains true, if everywhere the word *homeomorphism* is replaced by the word *diffeomorphism* and Homeo is replaced by Diffeo.

8.2. **Homotopy types of groups of homeomorphisms.** The group $\mathrm{Homeo}(S^1)$ contains $O(2)$ as a subgroup, which is its deformation retract. This follows from Lemma 8.3. More precisely, for each point $x \in S^1$, Lemma 8.3 provides a deformation retraction $\mathrm{Homeo}(S^1) \to O(2)$.

Similarly, the group of self-homeomorphisms of S^1 isotopic to the identity contains $SO(2) = S^1$ as a subgroup, which is its deformation retract.

The groups of self-homeomorphisms of $\mathbb{R}$, $\mathbb{R}_+$ and I which are isotopic to the identity are contractible. The contraction is provided by the rectilinear isotopy from Lemma 8.2 applied to $f = \mathrm{id}$ and an arbitrary g.

Thus for each connected 1-manifold X the group of homeomorphisms $X \to X$ isotopic to the identity is homotopy equivalent to X.

9. Finite group actions

9.1. The invariant intrinsic metric.

Theorem 9.1. *For any effective action of a finite group G on a 1-manifold X, there exists an intrinsic metric on X invariant under this action.*

Proof. Such a metric is created in three steps.

(1) Choose any metric d on X defining the topology.
(2) Symmetrize d: define for any $x, y \in X$ the number $s(x, y) = \sum_{g \in G} d(g(a), g(b))$. This is a metric and it is invariant under the action.
(3) Take the intrinsic metric s_I induced by s. □

Since an intrinsic metric on a 1-manifold defines an isometry onto one of the standard models, this gives a equivariant homeomorphism to one of the standard connected 1-manifolds with intrinsic metric and action of G by isometries.

9.2. Free actions.

If the action of a finite group G on a 1-manifold is free, then the orbit space X/G is a 1-manifold and the natural projection $X \to X/G$ is a covering.

Therefore the theory of coverings gives a simple classification of free finite group actions on 1-manifolds.

A contractible 1-manifold has no non-trivial covering. Thus, if a free finite group action on 1-manifold X has a contractible orbit space $Y = X/G$, then X is a disjoint union of copies of Y and G permutes these copies. In particular, there is no non-trivial free group action on a *connected* 1-manifold having contractible orbit space.

Coverings $X \to S^1$ with connected X are in one-to-one correspondence with subgroups of finite indices of $\pi_1(S^1) = \mathbb{Z}$. For each $m \in \mathbb{Z}$ there is one subgroup with index m, and hence one m-fold covering. The total space is homeomorphic to S^1, and the covering is equivalent to $S^1 \to S^1 : z \mapsto z^m$. In the corresponding action, the group is cyclic of order m and acts on S^1 by rotations.

In this classification of free finite group actions on connected 1-manifolds, the orbit space plays the main role. However, it is easy to reformulate it with emphasis on the 1-manifold on which the group acts. This is done in the next two theorems.

Theorem 9.2. *There is no non-trivial free finite group action on a contractible 1-manifold.*

Theorem 9.3. *If a finite group G acts freely on the circle than G is cyclic. Any finite cyclic group has a linear free action on S^1. Any free action of a finite cyclic group on S^1 is conjugate to a linear action.*

9.3. Asymmetry of a half-line.

Theorem 9.4. *There is no non-trivial action of a finite group on $\mathbb{R}_+$.*

Proof. We will prove that the only homeomorphism $h : \mathbb{R}_+ \to \mathbb{R}_+$ of finite order is the identity. Observe first that any homeomorphism $\mathbb{R}_+ \to \mathbb{R}_+$ must map a boundary point to a boundary point, and hence it preserves the only boundary point $0 \in \mathbb{R}_+$. By Theorem 9.1, there exists an intrinsic metric invariant under h. Therefore, h preserves the distance from the boundary point. However each point on $\mathbb{R}_+$ is uniquely defined by its distance to the boundary point.

□

9.4. **Actions on line and segment.**

Theorem 9.5. *The only orientation preserving homeomorphism $h : \mathbb{R} \to \mathbb{R}$ of finite order is the identity.*

Proof. Fix a homeomorphism $f : (0,\infty) \to \mathbb{R}$ (say, define it by formula $f : x \mapsto (x^2-1)/x$). Consider a homeomorphism $(0,\infty) \to (0,\infty) : x \mapsto f^{-1}hf(x)$. It preserves orientation (since h preserves orientation). So, it is a monotone increasing bijection $(0,\infty) \to (0,\infty)$ of finite order. Since it is increasing, it can be extended to $\mathbb{R}_+$ by letting $0 \mapsto 0$. The extended homeomorphism has the same finite order. But by Theorem 9.4 any such homeomorphism is the identity. □

Theorem 9.6. *Any orientation reversing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ of finite order has order two. It is conjugate to the reflection in a point.*

Proof. An orientation reversing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ is a monotone decreasing bijection. Consider the function $x \mapsto h(x) - x$. It is also a monotone decreasing bijection $\mathbb{R} \to \mathbb{R}$ and hence there exists a unique $a \in \mathbb{R}$ such that $h(a) - a = 0$, that is $h(a) = a$.

The homeomorphism h maps each connected component of $\mathbb{R} \setminus a$ to a connected component of $\mathbb{R} \setminus a$. The connected components are open rays $(-\infty, a)$ and (a, ∞). If each of them is mapped to itself, then h defines a homeomorphism of a finite order of the closed rays $(-\infty, a]$ and $[a, \infty)$. Then by Theorem 9.4, h is the identity, which contradicts our assumption. Thus, $h([a,\infty)) = (-\infty, a]$ and $h((-\infty, a]) = [a, \infty)$. Then h^2 preserves the rays, and, by Theorem 9.4, is the identity. Thus h has order two.

Choose a homeomorphism $f : [0,\infty) \to [a,\infty)$. Define function $g : (-\infty, 0] \to (-\infty, a]$ by the formula $g(x) = hf(-x)$. It's a homeomorphism. Together, f and g form a homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$. As is easy to check, $\phi^{-1}h\phi(x) = -x$. □

Theorem 9.7. *A non-trivial finite group acting effectively on $\mathbb{R}$ is a cyclic group of order 2. The action of the non-unit element is conjugate to the reflection in a point.*

Proof. As follows from Corollaries 9.5 and 9.6, any non-trivial element of the group is an orientation reversing involution. We have to prove that the group contains at most one such element. Assume that there are two orientation reversing homeomorphisms, f and g of the line $\mathbb{R}$. Their composition $f \circ g$ preserves orientation. Since it belongs to a finite group, it has finite order. By Theorem 9.5, it is the identity. So, $fg = 1$ and hence $f = g^{-1}$. But $g^2 = 1$. Therefore $g^{-1} = g$ and $f = g^{-1} = g$. □

Corollary 9.8. *A non-trivial finite group acting effectively on I is a cyclic group of order 2. The action of the non-unit element is conjugate to the reflection in a point.*

Proof. Any auto-homeomorphism of I preserves the boundary and the interior of I. Hence an effective finite group action on I induces an action of a finite group on the interior of I. An auto-homeomorphism of I is recovered from its restriction to the interior. Moreover, any auto-homeomorphism of $\operatorname{Int} I$ has a unique extension to an auto-homeomorphism of I. The interior $\operatorname{Int} I$ is homeomorphic to $\mathbb{R}$. □

9.5. **Actions on the circle.**

Theorem 9.9. *Any periodic orientation reversing homeomorphism $S^1 \to S^1$ is an involution (i.e., has period 2). It is conjugate to a reflection of S^1 in its diameter.*

Proof. Observe first that any orientation reversing auto-homeomorphism of the circle has a fixed point. One can prove this by elementary arguments, but we just refer to the Lefschetz Fixed Point Theorem: the Lefschetz number of such a homeomorphism is 2.

Consider the complement of a fixed point. The restriction of the homeomorphism to this complement satisfies the conditions of Theorem 9.6, which gives the required result. □

Observe that by Theorems 9.6, 9.8, and 9.9 any non-identity periodic homeomorphism of a connected 1-manifold with a fixed point is an involution reversing orientation.

Theorem 9.10. *A periodic non-identity orientation preserving homeomorphism $S^1 \to S^1$ has no fixed point. It is conjugate to a rotation.*

Proof. If if had a fixed point, then we could consider its restriction to the complement of this point, and by Theorem 9.5 would conclude that it is the identity and hence the whole homeomorphism is the identity.

For the same reasons, the non-identity powers of our periodic non-identity orientation preserving homeomorphism $S^1 \to S^1$ have no fixed points. Therefore, these powers form a cyclic group freely acting on S^1. See Theorem 9.3. □

Theorem 9.11. *A finite group acting effectively on S^1 is either cyclic or dihedral, and the action is conjugate to a linear one and extends to the standard (linear) action of the orthogonal group $O(2)$. The standard actions of cyclic and dihedral groups on the circle are provided by the symmetry groups of regular polygons.*

Proof. By Theorem 9.1, for a finite group action on S^1 there exists an invariant intrinsic metric. By Theorem 6.1, with this metric S^1 is isometric to the circle defined in the plane $\mathbb{R}^2$ by an equation $x^2 + y^2 = r^2$. Any isometry of this circle belongs to $O(2)$.

If all the homeomorphisms in the action preserve orientation, then by Theorem 9.10 the action is free, and the result follows from Theorem 9.3.

Assume that the action contains an orientation reversing homeomorphism. Then the group acts on the set of orientations. The orientation preserving homeomorphisms form a subgroup of index two. This is a cyclic group as above. Its complement consists of orientation reversing involutions. If the subgroup of orientation preserving homeomorphisms is trivial, then the whole group is of order 2 and the only non-trivial element is an orientation reversing involution, as in Theorem 9.9 above.

By Theorem 9.1, the action is equivalent to an action consisting of isometries of the circle. Its orientation preserving part consists of rotations and is cyclic, say of order m. Its elements are rotations by angles $2\pi k/m$. The orientation reversing elements are reflections in diameters. There are m diameters.

When $m = 2$, that is the group contains only two orientation preserving homeomorphisms, the whole group is the cartesian product of two cyclic groups of order 2. It is called Klein's *Vierergruppe* or *dihedral* group D_2. It contains two reflections in diameters orthogonal to each other, the symmetry in the center of the circle and the identity.

If the number of orientation preserving homeomorphisms is $m > 2$, then the whole group is called the dihedral group D_m. It is the symmetry group of an m-sided regular polygon. □

10. Exotic relatives of 1-manifolds

As we eliminate the Hausdorff or second countability property, the theory becomes somehow weird, but many aspects survive.

10.1. **First examples of non-Hausdorff 1-manifolds.**

1. Line with two origins. In the disjoint union of two copies $\mathbb{R}$ identify each point of one of the copies different from the origin with its corresponding point from the second copy: $\mathbb{R} \times \{+1, -1\}/(x, +1) \sim (x, -1)$ unless $x = 0$. This space has a single point for each nonzero real number x and two points $(0, +1)$ and $(0, -1)$ taking place of the origin. Each neighborhood of $(0, +1)$ intersects each neighborhood of $(0, -1)$, so the space is not Hausdorff.

2. Branching line. This is also a quotient space of two copies of the real line: $\mathbb{R} \times \{-1, 1\}/(x, +1) \sim (x, -1)$ if $x < 0$. This space has a single point for each real number $x < 0$ and two points $(x, +1)$, $(x, -1)$ for every non-negative x. As in the line with two origins above, in this space there is only one pair of points that have no disjoint neighborhoods: $(0, +1)$ and $(0, -1)$.

10.2. **Spaces of leaves.** At first glance, the examples above of non-Hausdorff 1-manifolds look artificial. However they appear naturally in some classical mathematical contexts. For example, the branching line is homeomorphic to the space of leaves of the following foliation of the plane $\mathbb{R}^2$. The leaves are the vertical lines $x = a$ with $|a| \geq 1$ and the graphs $y = b + \frac{1}{x^2-1}$ with $-1 < x < +1$ and arbitrary real b. In other words, the branching line is homeomorphic to the quotient space of $\mathbb{R}^2$ by partition into these lines and graphs. The leaves $x = \pm a$ correspond to points $(x, +1)$, $(x, -1)$ for non-negative x, the leaves $y = b + \frac{1}{x^2-1}$ with $-1 < x < +1$ correspond to points $x < 0$.

Similarly, the line with two origins can be identified with the space of leaves of a foliation of the cylinder $S^1 \times \mathbb{R}$.

Although at first glance the space of leaves of a foliation looks substantially more natural than descriptions of non-Hausdorff 1-manifolds as quotient spaces of some 1-manifolds, they are not far away from each other: in either case we deal with factorization of a nice Hausdorff space, but factorization easily gives rise to a non-Hausdorff space.

10.3. **Uncountable family of non-homeomorphic connected non-Hausdorff 1-manifolds.** If we enlarge the collection of spaces by eliminating the Hausdorff property, then the number of topological types of connected spaces becomes uncountable.

Indeed, one can take the disjoint union of two copies of the line $\mathbb{R}$ and identify an open set in one of them with its copy in the other one by the identity map. The quotient space is connected and satisfies all the requirements from the definition of 1-manifold except the Hausdorff axiom. In this way one can construct uncountably many pairwise non-homeomorphic spaces. To prove that they are not homeomorphic, one can use, for example, the topological type of the subset formed by those points that do not separate the space.

10.4. **Non-orientable non-Hausdorff 1-manifolds.** The definition of orientation from Section 4.2 generalizes straightforwardly to non-Hausdorff 1-manifolds. Unlike usual 1-manifolds, there exist non-Hausdorff 1-manifolds that are non-orientable. For example, the quotient space of the open interval $(-2, 2)$ under identification $x \sim x + 3$ for $x \in (-2, -1)$ is a non-Hausdorff 1-manifold which does not admit any orientation.

10.5. **Differential structures.** The notion of differential structure has a natural generalization to non-Hausdorff 1-manifolds. Let X be a non-Hausdorff 1-manifold, that is a second countable topological space such that each point of X has a neighborhood homeomorphic either to $\mathbb{R}$ or $\mathbb{R}_+$. Let r be either a natural number, or ∞. A sheaf D of real valued functions on X is called a *differential structure of class C^r* or just *C^r-structure* if it satisfies the following condition: for any open set $U \subset X$, a section $g \in D(U)$ and a function $f : g(U) \to \mathbb{R}$ of class C^r (i.e., a function having the first r continuous derivatives $f,\ f', \dots, f^{(r)}$ if r is finite and derivatives of all natural orders if $r = \infty$), the composition $f \circ g$ belongs to $D(U)$.

If each point $a \in X$ has a neighborhood $U \subset X$ homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$ such that a C^r-structure D, as a sheaf, induces on U a sheaf isomorphic to the sheaf on $\mathbb{R}$ or $\mathbb{R}_+$ of functions of class C^r, then the C^r-structure D is called *non-singular*. A 1-manifold equipped with a non-singular C^r-structures is a C^r-manifold of dimension 1.

Any 1-manifold admits a non-singular C^r-structure as in Section 6.3, and any two homeomorphic C^r-manifolds of dimension 1 are diffeomorphic, see Section 6.3 above.

10.6. **Homeomorphic, but non-diffeomorphic non-Hausdorff 1-manifolds.** There exist two homeomorphic, but not diffeomorphic non-Hausdorff 1-manifolds. In order to construct such an example, take a pair of monotone decreasing sequences a_n and b_n on $\mathbb{R}$ convergent to 0. There exists a homeomorphism $h : \mathbb{R} \to \mathbb{R}$ with $h(a_n) = b_n$ for all n, but one can find sequences for which there is no diffeomorphism with this property. For example, this is the case for $a_n = \frac{1}{n}$ and $b_n = \frac{1}{n^2}$. Take such a pair of sequences. Let $A = \mathbb{R} \smallsetminus (0 \cup \{a_n \mid n \in \mathbb{N}\})$ and $B = \mathbb{R} \smallsetminus (0 \cup \{b_n \mid n \in \mathbb{N}\})$. By the identity map of A attach one copy of $\mathbb{R}$ to another and denote the result by X. Similarly, in the disjoint sum of two copies of $\mathbb{R}$ identify by the identity map the copies of B and denote the result by Y. The homeomorphism h defines a homeomorphism $X \to Y$, on the other hand there is no diffeomorphism between X and Y, because if one existed, it would map A to B (as the sets of separating points) and extend to a diffeomorphism of $\mathbb{R}$ mapping a_n to b_n.

10.7. **Relatives of 1-manifolds without countable base.** One can easily construct a topological space which satisfies all the conditions of the definition of a 1-manifold except second countability by taking the disjoint sum of an uncountable number of copies of $\mathbb{R}$ (or any other 1-manifold). There are more interesting examples, which are connected. Most famous of them is the long line.

10.8. **Sheaf of germs of functions on a 1-manifold.** The sheaf of germs of continuous functions on a 1-manifold is locally homeomorphic to $\mathbb{R}$ or $\mathbb{R}_+$. So, it satisfies one condition (out of three) of the definition of a 1-manifold. The sheaf of germs of differentiable functions on a 1-manifold has a natural differential structure.

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