Bulletin of the Manifold Atlas - definition (2013)

Orientation of manifolds - definition*

MATTHIAS KRECK

1. Zero dimensional manifolds

For **zero dimensional manifolds** an orientation is a map from the manifold to ± 1, i.e. an orientation is a map $\epsilon : M \to \{\pm 1\}$. From now on we assume that all manifolds have positive dimension. Unless otherwise stated the manifolds have empty boundary.

2. Orientation of topological manifolds

An orientation of a topological manifold is a choice of a maximal atlas, such that the coordinate changes are orientation preserving. To make this precise we have to define when a homeomorphism from an open subset U of $\mathbb{R}^n$ to another open subset V is orientation preserving. We do this in terms of singular homology groups.

Definition 2.1. A homeomorphism f from an open subset V of $\mathbb{R}^n$ to another open subset V' is orientation preserving, if for each $x \in V$ the map $H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \cong H_n(V, V-x; \mathbb{Z}) \xrightarrow{f_*} H_n(V', V'-f(x); \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ is the identity map. Here the isomorphisms $H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z}) \cong H_n(V, V-x)$ is the following: We first take the map to $H_n(\mathbb{R}^n, \mathbb{R}^n - x)$ (or to $H_n(\mathbb{R}^n, \mathbb{R}^n - x)$) induced by the translation mapping 0 to x resp. $f(x)$ and then the inverse of the excision isomorphism. The isomorphism $H_n(V', V' - f(x); \mathbb{Z}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ is the inverse of the corresponding map.

Definition 2.2. An **orientation of an n-dimensional topological manifold** M is the choice of a maximal **oriented atlas**. Here an atlas $\{(U_i, \varphi_i : U_i \to V_i \subset \mathbb{R}^n)\}$ is called *oriented* if all coordinate changes $\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are orientation preserving. An oriented atlas is called maximal if it cannot be enlarged to an oriented atlas by adding another chart.

Note that any oriented atlas defines a maximal oriented atlas by adding all charts such that the atlas is still oriented. This is normally the way an oriented atlas is given.

A topological manifold M is called **orientable** if it has a topological orientation, otherwise it is called **non-orientable**.

A topological manifold M together with a topological orientation is called an **oriented topological manifold**.

An open subset of an oriented topological manifold is oriented by restring the charts in the maximal oriented atlas to the intersection with the open subset. A homeomorphism $f : N \to M$ between oriented topological manifolds is **orientation**

*Atlas page :www.map.mpim-bonn.mpg.de/Orientation_of_manifolds

Accepted: 1st June 2013

preserving if for each chart $\varphi : U \to V \subset \mathbb{R}^n$ in the oriented atlas of N the chart φf is in the oriented atlas of M.

There are several equivalent formulations of orientations both for topological manifolds and for smooth manifolds which we will explain in the following sections.

3. Reformulation in terms of local homological orientations

An orientation of an n-dimensional topological manifold M can also be defined in terms of the **local homology groups** $H_n(M, M - x; \mathbb{Z})$ for each x in M. Recall that there is an isomorphism $H_n(M, M - x; \mathbb{Z}) \cong \mathbb{Z}$ [3, 22.1].

Definition 3.1. A **local homological orientation of an n-dimensional topological manifold** M is the choice of a generator $[M]_x$ of the local homology group $H_n(M, M - x; \mathbb{Z})$ for each $x \in M$. Such a choice is called **continuous**, if for each $x \in M$ there is an open neighborhood U and a class $\alpha \in H_n(M, M - U; \mathbb{Z})$ such that the map induced by the inclusion $(M, M - U) \to (M, M - x)$ maps α to $[M]_x$ for each $x \in U$. A **homological orientation** of M is a continuous choice of local homological orientations.

As above an open subset U of M has an induced homological orientation which is given by the image under the inverse of the isomorphism induced by the inclusion $H_n(U, U - x; \mathbb{Z}) \to H_n(M, M - x; \mathbb{Z})$.

To get an example consider a finite dimensional **oriented real vector space** V, i.e. V is equipped with an equivalence class of bases $v_1, ..., v_n$, where two bases are called equivalent, if and only if the matrix of the base change matrix has positive determinant. The orientation of V as a vector space gives a homological orientation of V as a topological space as follows. We first orient at 0 in V by considering the simplex spanned by $-\sum_{i=1,...,n} v_i, v_1, v_2, ..., v_n$. This contains 0 in its interior and is a generator of $H_n(V, V - 0; \mathbb{Z})$. By translations we define local orientations at arbitrary points of V mapping the local orientation at 0 to the local orientation at x by the map induced by the translation mapping 0 to x. By construction this is a continuous family of local homological orientations and so gives a homological orientation of V. From this we obtain homological orientations of all open subsets of V.

The **equivalence of these two concepts** of an orientation of a topological manifold is shown as follows. A homeomorphism between manifolds equipped with a continuous family of local orientations is called **orientation preserving** if the induced map maps the corresponding local orientations to each other. We note that if both manifolds are open subsets of $\mathbb{R}^n$, this definition of orientation preserving homeomorphisms agrees with the one defined above. With this one defines for a topological manifold with a continuous family of local orientations a maximal oriented atlas by all charts which are orientation preserving, where we orient $\mathbb{R}^n$ as above. In turn if one has a maximal oriented atlas one uses it to transport the local orientations of open subsets of $\mathbb{R}^n$ to local homological orientations of M, which are a continuous family, since the atlas is oriented.

4. Orientation of smooth manifolds

The definition of an orientation for a topological manifold needs homology groups. For smooth manifolds the definition can be simplified. To distinguish the very similar definition we call it a smooth orientation.

Definition 4.1. A **smooth orientation of an n-dimensional smooth manifold** M is the choice of a maximal smooth **oriented atlas**. A smooth atlas

$$\{(U_i, \varphi_i : U_i \to V_i \subset \mathbb{R}^n)\}$$

is called **oriented** if the determinant of the derivatives of all coordinate changes $\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ is positive. A smooth oriented atlas is called *maximal* if it cannot be enlarged to a smooth oriented atlas by adding another smooth chart. Note that any smooth oriented atlas defines a maximal smooth oriented atlas by adding all smooth charts such that the atlas is still oriented. This is normally the way a smooth oriented atlas is given.

A smooth manifold M is called **orientable** if it has a smooth orientation, otherwise it is called **non-orientable**.

A smooth manifold M together with a smooth orientation is called an **oriented smooth manifold**.

5. From smooth orientation to homological orientation

For smooth manifolds we have now two definitions of an orientation, the smooth orientation and the orientation as a topological manifold. Here we explain why they are again equivalent concepts. The key observation is the following. If we have an orientation of the vector space $\mathbb{R}^n$ we have defined corresponding local orientations. If we change the orientation of the vector space $\mathbb{R}^n$, the local homological orientation changes its sign. Since there are two orientations of $\mathbb{R}^n$ as a vector space and two generators of $H_n(\mathbb{R}^n, \mathbb{R}^n - 0; \mathbb{Z})$ this correspondence is a bijection.

The next observation is that if we have a diffeomorphism from an open subset V in $\mathbb{R}^n$ to another open subset V' in $\mathbb{R}^n$, then its differential preserves the standard orientation of $\mathbb{R}^n$ if and only if it preserves the corresponding local homological orientations and so the underlying homeomorphism is orientation preserving in the sense defined in the beginning of the last section.

Thus, if M is smoothly oriented, i.e. is equipped with a maximal oriented smooth atlas, then - forgetting the smooth structure - we obtain an oriented topological atlas and we define the corresponding topological orientation by passing to the maximal oriented topological atlas containing these charts. In turn, if one has a maximal oriented topological atlas the subset of smooth charts in it defines a smooth orientation.

6. Reformulations of orientation for smooth manifolds

There are several equivalent formulations for orientations of smooth manifolds.

Definition 6.1. A **tangential orientation** of M is a **continuous choice** of an orientation of the tangent space T_xM in the sense of orientations of vector spaces for every point $x \in M$. Here continuous means that for every $x \in$M there is a chart

$\varphi : U \to V \subset R^n$ around x, such that the differential of φ maps for all $y \in U$ the orientation at T_yM to the same orientation of $T_{\varphi(y)}V = \mathbb{R}^n$.

The relationship between Definition 6.1 and Definition 4.1 is the following. If $\{(U_i, \varphi_i : U_i \to V_i \subset \mathbb{R}^n)\}$ is an oriented atlas, we define an orientation of T_xM by choosing an oriented chart $\varphi : U \to V \subset R^n$ around x and define the tangential orientation as the image of the orientation of $\mathbb{R}^n = T_{\varphi(x)}\mathbb{R}^n$ under the differential of φ^{-1}. In turn, if a continuous orientation of T_xM for all $x \in M$ is given, one defines a maximal oriented atlas as the atlas consisting of all charts $\varphi : U \to V \subset \mathbb{R}^n$ such that all differentials are orientation preserving, where we equip V with the induced orientation from $\mathbb{R}^n$ equipped with the standard orientation given by the canonical basis. It is easy to check that these constructions are well defined and give equivalent formulations.

Further equivalent formulations, which need a bit more knowledge of vector bundles are:

- An orientation of a smooth n-dimensional manifold M is given by the **reduction of the structure group** $GL_n(\mathbb{R}^n)$ of the tangent bundle TM to $GL_n(\mathbb{R}^n)^+$, the subgroup of matrices with determinant > 0. That this is an equivalence is an easy exercise.

- An orientation of a smooth manifold is given by a **trivialization** (an isomorphism to the trivial bundle) of the exterior bundle $\Lambda^n TM$. That this is an equivalence is an easy exercise.

Remark 6.2. Since the different concepts of orientations are all equivalent, one normally speaks of an oriented manifold in all cases. We only used the adjectives to make clear that a priory the definitions are different.

7. Criteria for orientability

There are various criteria for orientability:

Theorem 7.1. *A smooth n-dimensional manifold is orientable if and only if the tangent bundle (or the normal bundle of an embedding into $\mathbb{R}^k$) has a Thom class, i.e. a class $U \in H^n(TM, TM - 0; \mathbb{Z})$, whose restriction to each fibre TM_x is a generator of $H^n(TM_x, TM_x - 0; \mathbb{Z})$. Moreover the choice of a Thom class determines an orientation and vice versa.*

Proof. The class U orients each fibre TM_x and hence defines an orientation of TM as in Definition 2.2. The converse if proven in, for example, [4, Theorem 10.4] and [1, Theorem 11.3]. □

Theorem 7.2. *A smooth manifold M is orientable if and only if the first Stiefel Whitney class of its tangent bundle vanishes. See* [4, Lemma 11.6 and Problem 12-A] *and* [1, Proposition 17.2].

Theorem 7.3 ([2, VIII Corollary 3.4])**.** *A connected closed n-dimensional manifolds M is orientable if and only if $H_n(M; \mathbb{Z})$ is non-zero, in which case it is isomorphic to $\mathbb{Z}$. The choice of a generator is called a* **fundamental class** $[M] \in H_n(M; \mathbb{Z})$*. The choice of a generator corresponds to the choice of an orientation* [2, VIII Definition 4.1]*. For a not necessarily connected compact oriented manifold M the components* *are oriented and the sum of the fundamental classes of the components define the fundamental class of M.*

There is a generalization of Theorem 7.3 to non-compact manifolds.

Theorem 7.4 ([3, Corollary 22.26])**.** *If M is arbitrary, then M is orientable if and only if for each compact connected subset $K \subset M$ there is a class $[M]_K \in H_n(M, M - K; \mathbb{Z})$, such that for each $x \in K$ the map induced by the inclusion maps $[M]_K$ to a generator of $H_n(M, M - x; \mathbb{Z})$ and the classes mapped to each other under the maps induced by the inclusion $H_n(M, M - K; \mathbb{Z}) \to H_n(M, M - K'; \mathbb{Z})$ for all $K' \subset K$. The images of the classes $[M]_K$ in $H_n(M, M - x; \mathbb{Z})$ define a homological orientation of M and in turn a homological orientation determines the classes $[M]_K$.*

8. Manifolds with boundary

For manifolds W with boundary an orientation is defined as an orientation of its interior. An orientation of W induces an orientation on the boundary ∂W. If W is 1-dimensional, we orient the boundary, which is 0-dimensional by attaching to $x \in \partial W$ the local orientation $\epsilon(x) = -1$, if the restriction of a chart around x from U to $V \subset \mathbb{R}_{>0}$ to the interior is in the oriented atlas of the interior of W. Otherwise we define $\epsilon(x) = +1$. For example it we orient the interval $[0, 1]$ by the atlas of the interior given by the identity map, then $\epsilon(0) = -1$ and and $\epsilon(1) = 1$.

If the dimension of W is positive, we define the induced orientation both for smooth or topological manifolds in terms of an induced maximal oriented atlas of the boundary. If $\varphi : U \to V \subset \mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_1 \geq 0\}$ is a (smooth) chart around a boundary point $x \in \partial W$, such that its restriction to the inner is in the oriented atlas of the inner of W, then the restriction of this chart to $U \cap \partial W$ is a chart of ∂W and these charts form a maximal oriented (smooth) atlas of $-\partial W$. The orientation given by this atlas is called the **induced orientation on** ∂W.

The convention, that we consider the negative orientation on the boundary is for smooth manifolds equivalent to choosing an identification of the restriction of the tangent bundle of W to ∂W with $\mathbb{R} \oplus T\partial W$, where we identify $\mathbb{R}$ with a subbundle by selecting the outward normal vector field". With other words for smooth manifolds the induced orientation is characterized as the orientation of $T_x\partial W$, such that any outward pointing normal vector plus this orientation is the given orientation of W.

As for compact manifolds W without boundary one can see that a compact connected manifold with boundary is orientable if and only if $H_n(W; \partial W; \mathbb{Z})$ is non-zero, in which case it is again isomorphic to $\mathbb{Z}$, [2]. The choice of a generator is called a **relative fundamental class** and again this fixes an orientation of W.

Our at the first glance slightly ad libitum looking convention is made in such a way that the following holds:

Theorem 8.1. *Let W be a compact oriented n-dimensional manifold with boundary. If $[W, \partial W] \in H_n(W, \partial W; \mathbb{Z})$ is the fundamental class compatible with the orientation, then $\partial([W, \partial W]) \in H_{n-1}(\partial W; \mathbb{Z})$ is the the fundamental class compatible with the induced orientation of the boundary as defined above.*

Since the proof of this result is not in standard text books (to my knowledge), we give it here.

Proof. The case of 1-dimensional manifolds is easy. Thus we assume that W has dimension > 1. Since the orientation is given locally (we use the homological formulation) it is enough to show that if we consider the local orientation of W in a chart near the boundary, the boundary operator maps it to the local orientation of ∂W in the restriction of this chart to the boundary. Here we choose the chart in such a way, that the orientation of W corresponds to the standard orientation of H^n (if not change your atlas by a reflection in H^n).

Thus we consider $H^n = \{x \in \mathbb{R}^n | \, x_1 \geq 0\}$ and the local orientation given by the standard basis of $\mathbb{R}^n$. Since we work with the half space we map the simplex constructed by the standard basis with edges $-\sum_{i=1,...,n} e_i, e_1, e_2, ..., e_n$, so that it is spanned instead by $e_0 := -\sum_{i=2,...,n} e_i, e_1, ..., e_n$. We denote this simplex by Δ. The class represented by this simplex in $H_n(H^n, H^n - y; \mathbb{Z})$ for some y in the inner of the simplex is the same as that of $\mathbb{R}^n$ given by the standard orientation of $\mathbb{R}^n$. If we begin with the fundamental class $[W, \partial W]$, consider its image under the boundary operator in $H_{n-1}(\partial W; \mathbb{Z})$ and pass to the local orientation at $0 \in \{0\} \times \mathbb{R}^{n-1}$, then it is represented by the restriction of $d(\Delta)$ to the boundary of H^n. More precisely $d(\Delta) = \sum_{i=0...,n} (-1)^i (< e_0, ...e_{i-1}, e_{i+1}, ..., e_n >$, where $< ... >$ corresponds to the simplex spanned by the corresponding vectors, and the local orientation in ∂H at 0 corresponding to the image of the fundamental class in $H_{n-1}(\partial W; \mathbb{Z})$ is given by $(-1)^1 < e_0, e_2, ..., e_n >= - < e_0, ..., e_n >$. But this is the negative of the local orientation of $\partial H^n = \mathbb{R}^{n-1}$ given by the standard basis. This finishes the proof and explains why we took the negative orientation in our construction of the induced orientation in terms of an atlas. □

9. Orientation of products

Given two oriented manifolds there is an obvious way to orient their product by choosing the product atlas. If M is smooth and we have given orientations as tangential orientations, we note that $T_{(x,y)}(M \times N)$ is isomorphic to $T_x(M) \oplus T_y(N)$ and the isomorphism is induced by the differential of the projections and then the product orientation is given by the juxtaposition of the orientations of $T_x(M)$ and $T_y(N)$.

Similarly if M^m and N^n are oriented by a continuos family of local homological orientations, we note that $H_{m+n}(M \times N, M \times N - (x, y); \mathbb{Z})$ is isomorphic to $H_m(M, M - x; \mathbb{Z}) \otimes H_n(N, N - y; \mathbb{Z})$, this isomorphism from the latter to the first is given by the cross product

$$H_n(M, M{-}x; \mathbb{Z}) \otimes H_n(N, N{-}y; \mathbb{Z}) \longrightarrow$$
$$H_{m+n}(M {\times} N, (M {\times} (N{-}y)) \cup ((M{-}x) {\times} N; \mathbb{Z})) = H_{m+n}(M {\times} N, M {\times} N{-}(x, y); \mathbb{Z}).$$

By definition of the cross product of the local homological orientation given by the standard basis of $\mathbb{R}^m$ with the local homological orientation given by the standard basis of $\mathbb{R}^n$ is the local homological orientation given by the standard basis of $\mathbb{R}^{m+n}$. Thus the different concepts of product orientations given by the product of an atlas and by the product of local homological orientations agree also.

As a consequence for compact oriented manifolds equipped with fundamental classes the cross product of the fundamental classes corresponds to the product of the orientations induced by the fundamental classes.

10. Orientation of complex manifolds

An n-**dimensional complex manifold** is a topological manifold together with an atlas $(U_i, \varphi_i : U_i \to V_i \subset \mathbb{C}^n)$ such that the coordinate changes are holomorphic maps. Given such an atlas the charts considered as maps to $\mathbb{R}^{2n}$ have orientation preserving coordinate changes, since a complex matrix considered as a real matrix has determinant > 0, the square of the norm of the complex determinant. Thus a complex manifold considered as a real manifold has this way a canonical orientation.

References

[1] G. E. Bredon, *Topology and geometry*, Graduate Texts in Mathematics 139, Springer-Verlag, 1993. MR 1224675 Zbl 0934.55001

[2] A. Dold, *Lectures on algebraic topology*, Springer-Verlag, 1995. MR 1335915 Zbl 0872.55001

[3] M. J. Greenberg and J. R. Harper, *Algebraic topology*, Benjamin/Cummings Publishing Co. Inc. Advanced Book Program, 1981. MR 643101 Zbl 0498.55001

[4] J. W. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton, N. J., 1974. MR 0440554 Zbl 1079.57504

Matthias Kreck
Endenicher Allee 60
D-53115 Bonn, Germany

E-mail address: kreck@math.uni-bonn.de
Web address: http://him.uni-bonn.de/homepages/prof-dr-matthias-kreck/