Bulletin of the Manifold Atlas - definition (2013)

Microbundle - definition*

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1. Definition

The concept of a **microbundle** of dimension n was first introduced in [3] to give a model for the tangent bundle of an n-dimensional topological manifold. Later Kister [2], and independently Mazur, showed that every microbundle uniquely determines a topological $\mathbb{R}^n$-bundle; i.e. a fibre bundle with structure group the homeomorphisms of $\mathbb{R}^n$ fixing 0.

Definition 1.1 ([3])**.** Let B be a topological space. An **n-dimensional microbundle** over B is a quadruple (E, B, i, j) where E is a space, i and j are maps fitting into the following diagram

$$B \xrightarrow{i} E \xrightarrow{j} B$$

and the following conditions hold:

1. $j \circ i = \mathrm{id}_B$.
2. For all $x \in B$ there exist open neigbourhood $U \subset B$, an open neighbourhood $V \subset E$ of $i(b)$ and a homeomorphism
$$h\colon V \to U \times \mathbb{R}^n$$

which makes the following diagram commute:

$$\begin{tikzcd}
& V \arrow[dd, "h"] \arrow[dr, "j|_V"] & \\
U \arrow[ur, "i|_U"] \arrow[dr, "\times 0"'] & & U \\
& U \times \mathbb{R}^n \arrow[ur, "p_1"'] &
\end{tikzcd} \quad .$$

The space E is called the **total space** of the bundle and B the **base space**.

Two microbundles (E_n, B, i_n, j_n), $n = 1, 2$ over the same space B are **isomorphic** if there exist neighbourhoods $V_1 \subset E_1$ of $i_1(B)$ and $V_2 \subset E_2$ of $i_2(B)$ and a homeomorphism $H\colon V_1 \to V_2$ making the following diagram commute:

$$\begin{tikzcd}
& V_1 \arrow[dd, "H"] \arrow[dr, "j_1|_{V_1}"] & \\
B \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & & B \\
& V_2 \arrow[ur, "j_2|_{V_2}"'] &
\end{tikzcd}$$

*Atlas page: www.map.mpim-bonn.mpg.de/Microbundle

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2. Examples

An important example of a microbundle is the **tangent microbundle** of a topological (or similarly PL) manifold M. Let

$$\Delta_M \colon M \to M \times M, \quad x \mapsto (x, x)$$

be the diagonal map for M.

Example 2.1 ([3, Lemma 2.1])**.** Let M be topological (or PL) n-manifold, and let $p_1 \colon M \times M \to M$ be the projection onto the first factor. Then

$$(M \times M, M, \Delta_M, p_1)$$

is an n-dimensional microbundle, the **tangent microbundle** τ_M of M.

Remark 2.2. An atlas of M gives a product atlas of $M \times M$ which shows that the second condition of a microbundle is fulfilled. Actually the definition of the micro tangent bundle looks a bit more like a normal bundle to the diagonal, a view which fits to the fact that the normal bundle of the diagonal of a smooth manifold M in $M \times M$ is isomorphic to its tangent bundle.

Another important example of a microbundle is the micro-bundle defined by a topological topological $\mathbb{R}^n$-bundle.

Example 2.3. Let $\pi \colon E \to B$ be a topological $\mathbb{R}^n$-bundle with zero section $s \colon B \to E$. Then the quadruple

$$(E, B, s, \pi)$$

is an n-dimensional microbundle.

3. The Kister-Mazur Theorem

A fundamental fact about microbundles is the following theorem, often called the Kister-Mazur theorem, proven independently by Kister and Mazur.

Theorem 3.1 ([2, Theorem 2])**.** *Let (E, B, i, j) be an n-dimensional microbundle over a locally finite, finite dimensional simplicial complex B. Then there is a neighbourhood of $i(B)$, $E_1 \subset E$ such that the following hold.*

1. *E_1 is the total space of a topological $\mathbb{R}^n$-bundle over B.*
2. *$(E_1, B, i, j|_{E_1})$ is a microbundle and the the inclusion $E_1 \to E$ is a microbundle isomorphism.*
3. *If $E_2 \subset E$ is any other such neighbourhood of $i(B)$ then there is a $\mathbb{R}^n$-bundle isomorphism $(E_1, B, i, j|_{E_1}) \cong (E_2, B, i, j|_{E_2})$.*

Remark 3.2. Microbundle theory is an important part of the work by Kirby and Siebenmann [1] on smooth structures and PL-structures on higher dimensional topological manifolds.

REFERENCES

[1] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Princeton University Press, Princeton, N.J., 1977. MR 0645390 Zbl 0361.57004

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[3] J. Milnor, *Microbundles. I*, Topology **3** (1964), no.suppl. 1, 53-80. MR 0161346 Zbl 0124.38404

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