1. Definition

The concept of a microbundle of dimension $n$ was first introduced in [3] to give a model for the tangent bundle of an $n$-dimensional topological manifold. Later Kister [2], and independently Mazur, showed that every microbundle uniquely determines a topological $\mathbb{R}^n$-bundle; i.e. a fibre bundle with structure group the homeomorphisms of $\mathbb{R}^n$ fixing 0.

**Definition 1.1** ([3]). Let $B$ be a topological space. An $n$-dimensional microbundle over $B$ is a quadruple $(E, B, i, j)$ where $E$ is a space, $i$ and $j$ are maps fitting into the following diagram

$$
\begin{array}{ccc}
B & \xrightarrow{i} & E \\
\downarrow{j} & & \downarrow{j} \\
B & \xrightarrow{i} & B
\end{array}
$$

and the following conditions hold:

1. $j \circ i = \text{id}_B$.
2. For all $x \in B$ there exist open neighbourhood $U \subset B$, an open neighbourhood $V \subset E$ of $i(b)$ and a homeomorphism $h : V \to U \times \mathbb{R}^n$

which makes the following diagram commute:

$$
\begin{array}{ccc}
V & \xrightarrow{j} & U \\
\downarrow{h} & & \downarrow{h} \\
U & \xrightarrow{i} & U \\
\end{array}
$$

The space $E$ is called the **total space** of the bundle and $B$ the **base space**.

Two microbundles $(E_n, B, i_n, j_n)$, $n = 1, 2$ over the same space $B$ are **isomorphic** if there exist neighbourhoods $V_1 \subset E_1$ of $i_1(B)$ and $V_2 \subset E_2$ of $i_2(B)$ and a homeomorphism $H : V_1 \to V_2$ making the following diagram commute:

$$
\begin{array}{ccc}
V_1 & \xrightarrow{j_1} & V_2 \\
\downarrow{H} & & \downarrow{H} \\
B & \xrightarrow{i_2} & B \\
\end{array}
$$

*Atlas page :www.map.mpim-bonn.mpg.de/Microbundle

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2. Examples

An important example of a microbundle is the **tangent microbundle** of a topological (or similarly PL) manifold $M$. Let

$$\Delta_M : M \to M \times M, \quad x \mapsto (x, x)$$

be the diagonal map for $M$.

**Example 2.1** ([3, Lemma 2.1]). Let $M$ be topological (or PL) $n$-manifold, and let $p_1 : M \times M \to M$ be the projection onto the first factor. Then

$$(M \times M, M, \Delta_M, p_1)$$

is an $n$-dimensional microbundle, the **tangent microbundle** $\tau_M$ of $M$.

**Remark 2.2.** An atlas of $M$ gives a product atlas of $M \times M$ which shows that the second condition of a microbundle is fulfilled. Actually the definition of the micro tangent bundle looks a bit more like a normal bundle to the diagonal, a view which fits to the fact that the normal bundle of the diagonal of a smooth manifold $M$ in $M \times M$ is isomorphic to its tangent bundle.

Another important example of a microbundle is the micro-bundle defined by a topological topological $\mathbb{R}^n$-bundle.

**Example 2.3.** Let $\pi : E \to B$ be a topological $\mathbb{R}^n$-bundle with zero section $s : B \to E$. Then the quadruple

$$(E, B, s, \pi)$$

is an $n$-dimensional microbundle.

3. The Kister-Mazur Theorem

A fundamental fact about microbundles is the following theorem, often called the Kister-Mazur theorem, proven independently by Kister and Mazur.

**Theorem 3.1** ([2, Theorem 2]). Let $(E, B, i, j)$ be an $n$-dimensional microbundle over a locally finite, finite dimensional simplicial complex $B$. Then there is a neighbourhood of $i(B)$, $E_1 \subset E$ such that the following hold.

1. $E_1$ is the total space of a topological $\mathbb{R}^n$-bundle over $B$.
2. $(E_1, B, i, j|_{E_1})$ is a microbundle and the the inclusion $E_1 \to E$ is a microbundle isomorphism.
3. If $E_2 \subset E$ is any other such neighbourhood of $i(B)$ then there is a $\mathbb{R}^n$-bundle isomorphism $(E_1, B, i, j|_{E_1}) \cong (E_2, B, i, j|_{E_2})$.

**Remark 3.2.** Microbundle theory is an important part of the work by Kirby and Siebenmann [1] on smooth structures and PL-structures on higher dimensional topological manifolds.

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References


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