**Immersion - definition**

**ULRICH KOSCHORKE**

1. **Definition**

We work in a fixed category CAT of topological, piecewise linear, $C^r$-differentiable ($1 \leq r \leq \infty$) or real analytic manifolds (second countable, Hausdorff, without boundary) and maps between them. $\mathring{B}^k$ denotes the open unit ball in $\mathbb{R}^k, k = 0, 1, \ldots$.

Let $f : M^m \rightarrow N^n$ be such a map between manifolds of the indicated dimensions $m \leq n$.

**Definition 1.1.** $f$ is a local immersion at a point $x \in M$ if there exist open neighbourhoods $U$ of $x$ and $V$ of $f(x)$ in $M$ and $N$, resp., such that $f(U) \subset V$ and:

1. there is a CAT-isomorphism $h : V \rightarrow \mathring{B}^n$ (i.e. both $h$ and $h^{-1}$ are CAT-maps) which maps $f(U)$ onto $\mathring{B}^n \cap (\mathbb{R}^m \times \{0\}) = \mathring{B}^m$; and
2. $h \circ f$ yields a CAT-isomorphism from $U$ onto $\mathring{B}^m$.

We call $f$ an immersion (and we write $f : M \leftrightarrow N$) if $f$ is a local immersion at every point $x \in M$.

Thus an immersion looks locally like the inclusion $\mathbb{R}^m \subset \mathbb{R}^n$ of Euclidean spaces. It allows us to visualize a given manifold $M$ in a possibly more familiar setting such as $N = \mathbb{R}^n$. E.g. the projective plane $\mathbb{R}P^2$ can be visualized in $\mathbb{R}^3$ with the help of the Boy’s surface, the image of a $C^\infty$-immersion: see for example the page on surfaces. The following two questions play an important role.

1. **Existence:** Given $M$ and $N$, is there any immersion $M \leftrightarrow N$ at all?
2. **Classification:** How many essentially different immersions exist?

2. **The smooth case**

This section is about the category of smooth, i.e. $C^\infty$, manifolds and maps. It follows from the inverse function theorem that a smooth map $f : M \rightarrow N$ between smooth manifolds is a local immersion at $x \in M$ precisely if the tangent map $(Tf)_x : T_xM \rightarrow Tf(x)(N)$ is injective. Thus $f$ is a smooth immersion if and only if it induces a vector bundle monomorphism $Tf : TM \rightarrow TN$. E.g. the figure $\heartsuit$ cannot be the image of a smooth immersion, due to the two sharp corners which don’t allow a well-defined tangent line. However there exists a smooth immersion $f : S^1 \leftrightarrow \mathbb{R}^2$ with image the figure $\infty$.

*Atlas page: www.map.mpim-bonn.mpg.de/Immersion

Accepted: 15th April 2013
Theorem 2.1. [4, Remark], [7], Phillips 1967 If (i) $m < n$, or if (ii) $M$ is open and $m = n$, then the map

$$T : \text{Imm}(M,N) \to \text{Mono}(TM,TN), \ f \to Tf,$$

is a weak homotopy equivalence. Here the space, $\text{Imm}(M,N)$, of all smooth immersions $f : M \hookrightarrow N$ and $\text{Mono}(TM,TN)$, the space of all vector bundle monomorphisms $\varphi : TM \to TN$, are endowed respectively with the $C^\infty$-topology and the compact-open topology.

**Remark 2.3.** For a good exposition of Theorem 2.1 see [1, pp.87 and 93].

**Corollary 2.4.** Under the assumptions of theorem 2.1 there exists an immersion $f : M \hookrightarrow N$ if and only if there is a vector bundle monomorphism from the tangent bundle $TM$ of $M$ to $TN$. E. g. if $M$ is parallelizable (i. e. $TM \cong M \times \mathbb{R}^m$) then $M \hookrightarrow \mathbb{R}^{m+1}$.

**Theorem 2.5.** [10, Remark] If $m \geq 2$ then there exists an immersion $M^m \hookrightarrow \mathbb{R}^{2m-1}$ (E. g. any surface can be immersed into $\mathbb{R}^3$).

**Remark 2.6.** See also e.g. [1, p. 86ff].

Theorem 2.5 is best possible as long as we put no restrictions on $M$.

**Example 2.7.** The real projective space $\mathbb{RP}^m$ cannot be immersed into $\mathbb{R}^{2m-2}$ if $m = 2^k$. This follows from an easy calculation using Stiefel-Whitney classes: see, [6, Theorem 4,8].

**Definition 2.8.** Two immersions $f, g : M \hookrightarrow N$ are *regularly homotopic* if there exists a smooth map $F : M \times I \to N$ which with $f_t(x) := F(x,t)$ satisfies the following:

1. $f_0 = f, \ f_1 = g$;
2. $f_t$ is an immersion for all $t \in I$.

**Corollary 2.9.** Assume $m < n$. Two immersions $f, g : M \hookrightarrow N$ are regularly homotopic if and only if their tangent maps $Tf, Tg : TM \to TN$ are homotopic through vector bundle monomorphisms.

**Example 2.10** $(M = S^m, N = \mathbb{R}^n, [7])$. The regular homotopy classes of immersions $f : S^m \hookrightarrow \mathbb{R}^n, \ m < n$, are in one-to-one correspondence with the elements of the homotopy group $\pi_m(V_{n,m})$, where $V_{n,m}$ is the Stiefel manifold of $m$-frames in $\mathbb{R}^n$. In particular, all immersions $S^2 \hookrightarrow \mathbb{R}^3$ are regularly homotopic (since $\pi_2(V_{3,2}) = 0$). E. g. the standard inclusion $f_0 : S^2 \subset \mathbb{R}^3$ is regularly homotopic to $-f_0$; i. e. you can ’turn the sphere inside out’ in $\mathbb{R}^3$, with possible self-intersections but without creating any crease.

**Remark 2.11.** The Smale-Hirsch theorem makes existence and classification problems accessible to standard methods of algebraic topology such as classical obstruction theorem (cf. e.g. [9]), characteristic classes (cf. e.g. [6]), Postnikov towers, the singularity method (cf. e.g. [5]) etc.: see [8] for the state of the art in 1963.
3. Self intersections

It is a characteristic feature of immersions - as compared to embeddings - that r-tuple selfintersections may occur for some r ≥ 2, i.e. points in N which are the image of at least r distinct elements of M (e.g. the double point in the figure 8 immersion f : S¹ ↪ R² with image ∞). Generically the locus of r-tuple points of a smooth immersion f : Mᵐ ↪ Nⁿ is an immersed (n − r(n − m))-dimensional manifold in N. Its properties may yield a variety of interesting invariants which link immersions to other concepts of topology. E.g. let θ(f) denote the mod 2 number of (n + 1)-tuple points of a selftransverse immersion f : Mⁿ ↪ Rⁿ⁺¹.

Theorem 3.1. [2] Given a natural number n ≡ 1(4), there is an n-dimensional closed smooth manifold Mⁿ and an immersion f : Mⁿ ↪ Rⁿ⁺¹ satisfying θ(f) = 1 if and only if there exists a framed (n + 1)-dimensional manifold with Kervaire invariant 1.

According to [3] (and previous authors) this holds precisely when n = 1, 5, 13, 29, 61 or possibly 125. If n ≠ 1 and n = 1(4) the manifold M in question cannot be orientable (cf. [2]). Thus the figure 8 immersion f : S¹ ↪ R² plays a rather special role here.

References


Ulrich Koschorke

Department of Mathematics, ENC,
Universität Siegen,
D57068 Siegen, Germany

E-mail address: koschorke@mathematik.uni-siegen.de

Bulletin of the Manifold Atlas - definition 2013