Immersion - definition*

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1. Definition

We work in a fixed category CAT of topological, piecewise linear, C^r-differentiable ($1 \leq r \leq \infty$) or real analytic manifolds (second countable, Hausdorff, without boundary) and maps between them. $\mathring{B}^k$ denotes the open unit ball in $\mathbb{R}^k, k = 0, 1, \dots$.

Let $f\colon M^m \to N^n$ be such a map between manifolds of the indicated dimensions $m \leq n$.

Definition 1.1. f is a *local immersion at a point* $x \in M$ if there exist open neighbourhoods U of x and V of $f(x)$ in M and N, resp., such that $f(U) \subset V$ and:

(1) there is a CAT-isomorphism $h : V \to \mathring{B}^n$ (i.e. both h and h^{-1} are CAT-maps) which maps $f(U)$ onto $\mathring{B}^n \cap (\mathbb{R}^m \times \{0\}) = \mathring{B}^m$; and
(2) $h \circ f$ yields a CAT-isomorphism from U onto $\mathring{B}^m$.

We call f an *immersion* (and we write $f : M \looparrowright N$) if f is a local immersion at every point $x \in M$.

Thus an immersion looks locally like the inclusion $\mathbb{R}^m \subset \mathbb{R}^n$ of Euclidean spaces. It allows us to visualize a given manifold M in a possibly more familar setting such as $N = \mathbb{R}^n$. E.g. the projective plane $\mathbb{R}\mathrm{P}^2$ can be visualized in $\mathbb{R}^3$ with the help of the *Boy's surface*, the image of a C^∞-immersion: see for example the page on surfaces. The following two questions play an important role.

(1) *Existence*: Given M and N, is there any immersion $M \looparrowright N$ at all?
(2) *Classification*: How many *essentially different* immersions exist?

2. The smooth case

This section is about the category of smooth, i.e. C^∞, manifolds and maps. It follows from the inverse function theorem that a smooth map $f : M \to N$ between smooth manifolds is a local immersion at $x \in M$ precisely if the tangent map $(Tf)_x : T_xM \to T_{f(x)}(N)$ is injective. Thus f is a smooth immersion if and only if it induces a vector bundle *mono*morphism $Tf : TM \to TN$. E.g. the figure ♡ cannot be the image of a smooth immersion, due to the two sharp corners which don't allow a well-defined tangent line. However there exists a smooth immersion $f : S^1 \looparrowright \mathbb{R}^2$ with image the figure ∞.

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Theorem 2.1. [4, Remark], [7], *Phillips 1967 If (i) $m < n$, or if (ii) M is open and $m = n$, then the map*

$$T : \mathrm{Imm}(M, N) \to \mathrm{Mono}(TM, TN), \quad f \to Tf, \tag{2}$$

is a weak homotopy equivalence. Here the space, $\mathrm{Imm}(M, N)$, *of all smooth immersions $f : M \looparrowright N$ and* $\mathrm{Mono}(TM, TN)$, *the space of all vector bundle monomorphisms $\varphi : TM \to TN$, are endowed respectively with the C^∞-topology and the compact-open topology.*

Remark 2.3. For a good exposition of Theorem 2.1 see [1, pp.87 and 93].

Corollary 2.4. *Under the assumptions of theorem 2.1 there exists an immersion $f : M \looparrowright N$ if and only if there is a vector bundle monomorphism from the tangent bundle TM of M to TN. E. g. if M is parallelizable (i. e. $TM \cong M \times \mathbb{R}^m$) then $M \looparrowright \mathbb{R}^{m+1}$.*

Theorem 2.5. [10, Remark] *If $m \geq 2$ then there exists an immersion $M^m \looparrowright \mathbb{R}^{2m-1}$ (E. g. any surface can be immersed into $\mathbb{R}^3$).*

Remark 2.6. See also e.g. [1, p. 86ff].

Theorem 2.5 is best possible as long as we put no restrictions on M.

Example 2.7. The real projective space $\mathbb{R}\mathrm{P}^m$ cannot be immersed into $\mathbb{R}^{2m-2}$ if $m = 2^k$. This follows from an easy calculation using Stiefel-Whitney classes: see, [6, Theorem 4,8].

Definition 2.8. Two immersions $f, g : M \looparrowright N$ are *regularly homotopic* if there exists a *smooth map* $F : M \times I \to N$ which with $f_t(x) := F(x, t)$ satisfies the following:

(1) $f_0 = f, \quad f_1 = g$;
(2) f_t is a immersion for all $t \in I$.

Corollary 2.9. *Assume $m < n$. Two immersions $f, g : M \looparrowright N$ are regularly homotopic if and only if their tangent maps $Tf, Tg : TM \to TN$ are homotopic through vector bundle monomorphisms.*

Example 2.10 ($M = S^m, N = \mathbb{R}^n$, [7])**.** The regular homotopy classes of immersions $f : S^m \looparrowright \mathbb{R}^n$, $m < n$, are in one-to-one correspondance with the elements of the homotopy group $\pi_m(V_{n,m})$, where $V_{n,m}$ is the Stiefel manifold of m-frames in $\mathbb{R}^n$. In particular, all immersions $S^2 \looparrowright \mathbb{R}^3$ are regularly homotopic (since $\pi_2(V_{3,2}) = 0$). E. g. the standard inclusion $f_0 : S^2 \subset \mathbb{R}^3$ is regularly homotopic to $-f_0$; i. e. you can *'turn the sphere inside out'* in $\mathbb{R}^3$, with possible self-intersections but without creating any crease.

Remark 2.11. The Smale-Hirsch theorem makes existence and classification problems accessible to standard methods of algebraic topology such as classical obstruction theorem (cf. e.g. [9]), characteristic classes (cf. e.g. [6]), Postnikov towers, the singularity method (cf. e.g. [5]) etc.: see [8] for the state of the art in 1963.

3. Self intersections

It is a characteristic feature of immersions - as compared to embeddings - that r-tuple selfintersections may occur for some $r \geq 2$, i. e. points in N which are the image of at least r distinct elements of M (e. g. the double point in the figure 8 immersion $f : S^1 \looparrowright \mathbb{R}^2$ with image ∞). Generically the locus of r-tuple points of a smooth immersion $f : M^m \looparrowright N^n$ is an immersed $(n - r(n - m))$-dimensional manifold in N. Its properties may yield a variety of interesting invariants which link immersions to other concepts of topology. E. g. let $\theta(f)$ denote the mod 2 number of $(n + 1)$-tuple points of a selftransverse immersion $f : M^n \looparrowright \mathbb{R}^{n+1}$.

Theorem 3.1. [2] *Given a natural number $n \equiv 1(4)$, there is an n-dimensional closed smooth manifold M^n and an immersion $f : M^n \looparrowright \mathbb{R}^{n+1}$ satisfying $\theta(f) = 1$ if and only if there exists a framed $(n + 1)$-dimensional manifold with Kervaire invariant 1.*

According to [3] (and previous authors) this holds precisely when $n = 1, 5, 13, 29, 61$ or possibly 125. If $n \neq 1$ and $n = 1(4)$ the manifold M in question cannot be orientable (cf. [2]). Thus the figure 8 immersion $f : S^1 \looparrowright \mathbb{R}^2$ plays a rather special role here.

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