1. Definition

We work in a fixed category CAT of topological, piecewise linear, $C^r$-differentiable ($1 \leq r \leq \infty$) or real analytic manifolds (second countable, Hausdorff, without boundary) and maps between them. $\hat{B}^k$ denotes the open unit ball in $\mathbb{R}^k, k = 0, 1, \ldots$

Let $f: M^m \rightarrow N^n$ be such a map between manifolds of the indicated dimensions $m \leq n$.

**Definition 1.1.** $f$ is a local immersion at a point $x \in M$ if there exist open neighbourhoods $U$ of $x$ and $V$ of $f(x)$ in $M$ and $N$, resp., such that $f(U) \subset V$ and:

1. there is a CAT-isomorphism $h : V \rightarrow \hat{B}^n$ (i.e. both $h$ and $h^{-1}$ are CAT-maps) which maps $f(U)$ onto $\hat{B}^n \cap (\mathbb{R}^m \times \{0\}) = \hat{B}^m$; and
2. $h \circ f$ yields a CAT-isomorphism from $U$ onto $\hat{B}^m$.

We call $f$ an immersion (and we write $f : M \hookrightarrow N$) if $f$ is a local immersion at every point $x \in M$.

Thus an immersion looks locally like the inclusion $\mathbb{R}^m \subset \mathbb{R}^n$ of Euclidean spaces. It allows us to visualize a given manifold $M$ in a possibly more familiar setting such as $N = \mathbb{R}^n$. E.g. the projective plane $\mathbb{RP}^2$ can be visualized in $\mathbb{R}^3$ with the help of the Boy’s surface, the image of a $C^\infty$-immersion: see for example the page on surfaces. The following two questions play an important role.

1. **Existence**: Given $M$ and $N$, is there any immersion $M \hookrightarrow N$ at all?
2. **Classification**: How many essentially different immersions exist?

2. The smooth case

This section is about the category of smooth, i.e. $C^\infty$, manifolds and maps. It follows from the inverse function theorem that a smooth map $f : M \rightarrow N$ between smooth manifolds is a local immersion at $x \in M$ precisely if the tangent map $(Tf)_x : T_x M \rightarrow T_{f(x)}(N)$ is injective. Thus $f$ is a smooth immersion if and only if it induces a vector bundle monomorphism $Tf : TM \rightarrow TN$. E.g. the figure $\heartsuit$ cannot be the image of a smooth immersion, due to the two sharp corners which don’t allow a well-defined tangent line. However there exists a smooth immersion $f : S^1 \hookrightarrow \mathbb{R}^2$ with image the figure $\infty$.  

*Atlas page : www.map.mpim-bonn.mpg.de/Immerssion

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Theorem 2.1. [4, Remark], [7], Phillips 1967 If (i) $m < n$, or if (ii) $M$ is open and $m = n$, then the map
\begin{equation}
(2) \quad T : \text{Imm}(M, N) \to \text{Mono}(TM, TN), \quad f \to Tf,
\end{equation}
is a weak homotopy equivalence. Here the space, $\text{Imm}(M, N)$, of all smooth immersions $f : M \hookrightarrow N$ and $\text{Mono}(TM, TN)$, the space of all vector bundle monomorphisms $\varphi : TM \to TN$, are endowed respectively with the $C^\infty$-topology and the compact-open topology.

Remark 2.3. For a good exposition of Theorem 2.1 see [1, pp.87 and 93].

Corollary 2.4. Under the assumptions of theorem 2.1 there exists an immersion $f : M \hookrightarrow N$ if and only if there is a vector bundle monomorphism from the tangent bundle $TM$ of $M$ to $TN$. E. g. if $M$ is parallelizable (i. e. $TM \cong M \times \mathbb{R}^m$) then $M \hookrightarrow \mathbb{R}^{m+1}$.

Theorem 2.5. [10, Remark] If $m \geq 2$ then there exists an immersion $M^m \hookrightarrow \mathbb{R}^{2m-1}$ (E. g. any surface can be immersed into $\mathbb{R}^3$).

Remark 2.6. See also e.g. [1, p. 86ff].

Theorem 2.5 is best possible as long as we put no restrictions on $M$.

Example 2.7. The real projective space $\mathbb{R}P^m$ cannot be immersed into $\mathbb{R}^{2m-2}$ if $m = 2^k$. This follows from an easy calculation using Stiefel-Whitney classes: see, [6, Theorem 4.8].

Definition 2.8. Two immersions $f, g : M \hookrightarrow N$ are regularly homotopic if there exists a smooth map $F : M \times I \to N$ which with $f_t(x) := F(x,t)$ satisfies the following:
\begin{enumerate}
\item $f_0 = f$, \quad $f_1 = g$;
\item $f_t$ is an immersion for all $t \in I$.
\end{enumerate}

Corollary 2.9. Assume $m < n$. Two immersions $f, g : M \hookrightarrow N$ are regularly homotopic if and only if their tangent maps $Tf, Tg : TM \to TN$ are homotopic through vector bundle monomorphisms.

Example 2.10 ($M = S^m, N = \mathbb{R}^n$, [7]). The regular homotopy classes of immersions $f : S^m \hookrightarrow \mathbb{R}^n$, $m < n$, are in one-to-one correspondence with the elements of the homotopy group $\pi_m(V_{n,m})$, where $V_{n,m}$ is the Stiefel manifold of $m$-frames in $\mathbb{R}^n$. In particular, all immersions $S^2 \hookrightarrow \mathbb{R}^3$ are regularly homotopic (since $\pi_2(V_{3,2}) = 0$). E. g. the standard inclusion $f_0 : S^2 \subset \mathbb{R}^3$ is regularly homotopic to $-f_0$; i. e. you can 'turn the sphere inside out' in $\mathbb{R}^3$, with possible self-intersections but without creating any crease.

Remark 2.11. The Smale-Hirsch theorem makes existence and classification problems accessible to standard methods of algebraic topology such as classical obstruction theorem (cf. e. g. [9]), characteristic classes (cf. e. g. [6]), Postnikov towers, the singularity method (cf. e. g. [5]) etc.: see [8] for the state of the art in 1963.

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3. Self intersections

It is a characteristic feature of immersions - as compared to embeddings - that $r$-tuple selfintersections may occur for some $r \geq 2$, i.e. points in $N$ which are the image of at least $r$ distinct elements of $M$ (e.g. the double point in the figure 8 immersion $f : S^1 \hookrightarrow \mathbb{R}^2$ with image $\infty$). Generically the locus of $r$-tuple points of a smooth immersion $f : M^n \hookrightarrow N^n$ is an immersed $(n - r(n - m))$-dimensional manifold in $N$. Its properties may yield a variety of interesting invariants which link immersions to other concepts of topology. E.g. let $\theta(f)$ denote the mod 2 number of $(n + 1)$-tuple points of a selftransverse immersion $f : M^n \hookrightarrow \mathbb{R}^{n+1}$.

**Theorem 3.1.** [2] Given a natural number $n \equiv 1(4)$, there is an $n$-dimensional closed smooth manifold $M^n$ and an immersion $f : M^n \hookrightarrow \mathbb{R}^{n+1}$ satisfying $\theta(f) = 1$ if and only if there exists a framed $(n + 1)$-dimensional manifold with Kervaire invariant 1.

According to [3] (and previous authors) this holds precisely when $n = 1, 5, 13, 29, 61$ or possibly 125. If $n \neq 1$ and $n = 1(4)$ the manifold $M$ in question cannot be orientable (cf. [2]). Thus the figure 8 immersion $f : S^1 \hookrightarrow \mathbb{R}^2$ plays a rather special role here.

**References**


Ulrich Koschorke
Department of Mathematics, ENC,
Universität Siegen,
D57068 Siegen, Germany

E-mail address: koschorke@mathematik.uni-siegen.de

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