Bulletin of the Manifold Atlas - definition (2013)

Embedding - definition*

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1. Definition

We work in a fixed category CAT of topological, piecewise linear, C^r-differentiable $(1 \leq r \leq \infty)$ or real analytic manifolds (second countable, Hausdorff, without boundary) and maps between them.

Let $f : M^m \to N^n$ be such a map between manifolds of the indicated dimensions $1 \leq m < n$.

Definition 1.1. We call f an **embedding** (and we write $f : M \hookrightarrow N$) if f is an immersion which maps M homeomorphically onto its image.

It follows that an embedding cannot have selfintersections. But even an injective immersion need not be an embedding; e.g. the figure six 6 is the image of a smooth immersion but not of an embedding. Note that in the topological and piecewise linear categories,CAT = TOP or PL, our definition yields *locally flat* embeddings. In these categories there are other concepts of embeddings - e.g. wild embeddings - which are not locally flat: the condition of local flatness is implied by our definition of immersion. Embeddings (and immersions) into familiar target manifolds such as $\mathbb{R}^n$ may help to visualize abstractly defined manifolds. E.g. all smooth surfaces can be immersed into $\mathbb{R}^3$; but non-orientable surfaces (such as the projective plane and the Klein bottle) allow no embeddings into $\mathbb{R}^3$.

2. Existence of embeddings

Theorem 2.1 ([2]). *For every compact m–dimensional PL-manifold M there exists a PL–embedding $M \hookrightarrow \mathbb{R}^{2m}$.*

Remark 2.2. For a good exposition of Theorem 2.1 see also [4, p. 63].

Theorem 2.3 ([5]). *For every closed m–dimensional C^∞–manifold M there exists a C^∞–embedding $M \hookrightarrow \mathbb{R}^{2m}$.*

Remark 2.4. For more modern expositions see also [1, p. 67ff] and [3, 22.1].

Similar existence results for embeddings $M^m \hookrightarrow \mathbb{R}^N$ are valid also in the categories of real analytic maps and of isometrics (Nash) when $N \gg 2m$ is sufficiently high.

*Atlas page: www.map.mpim-bonn.mpg.de/Embedding

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3. Classification

In order to get a survey of all *essentially distinct* embeddings $f : M \hookrightarrow N$ it is meaningful to introduce equivalence relations such as (ambient) isotopy, concordance, bordism etc., and to aim at classifying embeddings accordingly. The difficulty of this task depends heavily on the choices of M and N and especially their dimensions: for more information please see the page on high codimension embeddings. Already for the most basic choices of M and N this may turn out to be a very difficult task. E.g. in the theory of knots (or links) where M is a 1-sphere (or a finite union of 1-spheres), and $N = \mathbb{R}^3$ the multitude of possible knotting and linking phenomena is just overwhelming. Even classifying links up to the very crude equivalence relation *link homotopy* is very far from having been achieved yet.

References

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