Bulletin of the Manifold Atlas (2013)

Projective plane: a history*

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Abstract. We present the early history of projective space in dimensions two and three from a topological point of view (in the last third of the 19th century). In particular we study different models of these intriguing manifolds and the question of their topological invariants.

01A55, 01A60

The projective plane played an important role in the early history of manifolds (that is here surfaces), the projective space is a very natural example of a closed 3-manifold. The question how those intriguing objects were defined and how they were understood is difficult and complex, many details may be found in [1, 550]).

The projective plane entered the history of topology with the idea to apply Riemann's tools to known surfaces as the algebraic ones. L. Schläfli used them to study surfaces of degree three, he got in contact with F. Klein by correspondence ([4]). Klein and Schläfli discussed the question of which type of connectivity is realized in non-orientable surfaces. Klein observed the basic fact that a projective straight line in the projective plane doesn't bound a region but it bounds if one runs through it twice ([4]). This is the phenomenon later called "torsion" by Poincaré. In this context Klein introduced his idea of a "Doppelfläche" (double surface) and the notion of "Einseitigkeit" (one-sidedness) or of "surface with invertible indicatrix". Thus Klein was ready to understand that the closed surfaces fall into two classes: the orientable and the non-orientable ones ([10, 164-170]). Of course non-orientability was known since Listing and Möbius discovered the Möbius strip; but nobody applied the concept to closed surfaces. Even Jordan's classification of those surfaces (1866) didn't pay attention to the non-orientable case (for more details on Jordan cf. [11]).

It was not easy to understand the projective plane from a topological point of view but Klein found a useful model for it (cf. Figure 1). The points of the bounding circle are identified by pairs of diametric points. The text of Klein's book published in 1927 ([5]) is much older; it goes back to a lecture course that Klein delivered in Göttingen during 1889-90. Klein's picture immediately suggests the idea that the projective plane is a sphere with a hole, the hole being closed by a Möbius strip (or - as it is later called - a crosscap). With the help of his model Klein also explains the reversibility of the indicatrix in the projective plane (cf. fig. Figure 2).

Klein's ideas were taken up by his pupil W. Dyck. In 1888 Dyck published a first long paper on topology; its principal theme was the classification of surfaces. Dyck was the first mathematician to present a complete list of closed surfaces including orientable and non-orientable surfaces as well. His main tool was a characteristic

*Atlas page: http://www.map.him.uni-bonn.de/

Keywords: projective plance, projective space, history of topology

Accepted: February 12th 2013

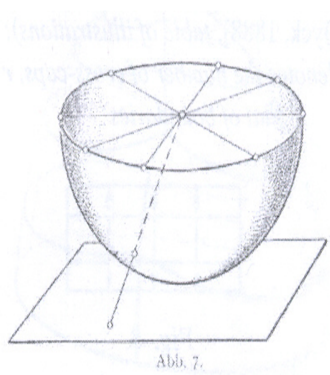

FIGURE 1. Klein's model of the projective plane[5, 14]

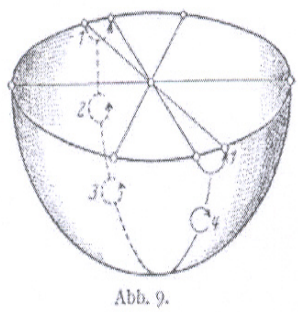

FIGURE 2. The little circle is reversed in travelling along a straight line in the projective plane [5, 15].

which he calculated with the help of a combinatorial structure on the surfaces. His main result was the following: Two surfaces are homeomorphic if and only if:

"1. The surfaces are members of the same class, that is, surfaces with a non-reversible indicatrix or surfaces with a reversible indicatrix. 2. Their characteristics are the same. 3. The number of boundaries is the same." ([2, 488])

For a part of Dyck's classification consider Figure 3.

The picture of the projective plane (in the first line of Figure 3 on the left) given by Dyck became standard - may be we replace the circle by a square. The other picture (first line on the right) is intended to illustrate the crosscap; K is the number of crosscaps, r the number of boundary curves. Let me remark that Dyck's proof of his main result is not complete - what is lacking is the demonstration of the fact that every surface can be transformed into one of the normal forms in Dyck's classification ([11, 60-72]). It should also be mentioned that Dyck realised the basic fact that two crosscaps can be transformed into one handle if at least one more cross caps stays. With Dyck's result the projective plane was integrated into a whole series of closed non-orientable surfaces; from a topological point of view it was well understood now.

From a topological point of view the case of the projective 3-space is much harder then that of the projective plane because the projective space is a 3-manifold. From a

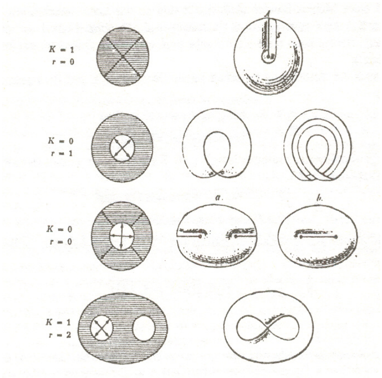

FIGURE 3. The first scheme yields the projective plane, the second the Möbius strip ([2, Dyck1888|table of illustrations], 488).

geometric point of view the difference between the two is not that big, even Desargues (1639) had no problem to work within projective space. The first contributions to the theory of 3-manifolds are due to W. Dyck. In his second - and last - long paper on topology he summarized his work on higher dimensional manifolds ([3], cf. [10]). "Projective manifolds" ("projektive Mannigfaltigkeiten" in Dyck's words) and the spheres are the two classes of examples of n-manifolds Dyck studied. There are two descriptions of projective manifolds $\mathbb{R}P^n$ (of dimension n which is 3 or higher), one by using homogenuous coordinates, the other by identifying diametric points on the sphere S^n. Dyck was able to calculate the n-th Betti number of $\mathbb{R}P^n$, which is 0 if n is odd and 1 if n is even. He concluded that $\mathbb{R}P^n$ is orientable if n is odd and non-orientable otherwise ([3]).

Projective space also showed up in Poincaré's paper on "Analysis situs" ([6, 233]) but not with that name. There is a series of examples of closed 3-manifolds constructed by Poincaré which serve as testing cases for his theory. Most of Poincaré's examples are obtained out of a cube by identifications on its faces. But the fifth example is different: Its starting point is a solid octahedron. the points of its faces are identified using the symmetry to the center of the octahedron ([9]). With the help of certain techniques based on his identifcation scheme Poincaré was able to calculate the fundamental group of his example - that is of $\mathbb{R}P^3$: It is a group containing only two elements, that is in modern notation $\mathbb{Z}_2$. Poincaré calculated also the first and the second Betti-number of his examples - both are 1. In this calculation he made a famous mistake in concluding that $2C \equiv 0$ implies $C \equiv 0$ (C is a 1-cycle) ([9, 245]). After the critique by Heegard (1898) Poincaré refined his theory in his first complement ([7, 353]) by distinguishing between homology with and homology without division (in this context he cited only his third example - the later so called quaternion space where he concluded $C \equiv 0$ from $4C \equiv 0$), in the second complement ([8]) he found an effective method to calculate torsion coefficients (using incidence matrices). In the case of his fifth example he found such a coefficient in dimension 1 with value 2 ([9]). So we may state that in 1900 Poincaré arrived at a complete knowledge of the standard topological invariants of 3-dimensional projective space!

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