Bulletin of the Manifold Atlas (2013)

Poincaré's cube manifolds*

KLAUS VOLKERT

Abstract. In this article we describe an important class of 3-manifolds studied by Poincaré in his papers on analysis situs (1892 – 1904). They are constructed by identifying the sides of a cube in more or less complicated ways and are therefore called here "cube manifolds". This idea goes back to Poincarés work on autormorphic functions (early 1880's).

01A55, 01A60

In his seminal paper "On Analysis situs", published in 1895 ([6, 288 - 289]), Poincaré constructed a series of closed 3-manifolds which played an important role in the development of his theory. They were both sources of inspiration and testing cases for the new tools. In particular Poincaré tested his topological invariants (the fundamental group, the Betti numbers) on them. Most of those examples are constructed by identifications on the faces of a (solid) cube. Therefore they are called here Poincaré's cube manifolds.

The simplest cube manifold is probably the 3-torus. It was described before Poincaré by W. Killing in the context of his work on Clifford-Klein space-forms ([3]). The faces of a cube are identified by three translations. Since these translations generate a subgroup of the isometries of ordinary 3-space which acts discontinuously and without fixed points, the resulting 3-torus - the name is neither used by Killing nor by Poincaré - is a locally flat 3D space-form.

In his work on Kleinean functions, which are special types of automorphic functions ([4, 274-279]), Poincaré had met the situation that the faces of a polyhedron are identified in pairs ([8]). The difference here is that those polyhedra were hyperbolic polyhedra whereas in his toplogical work the geometric situation is not taken into account. It should be mentioned that Poincaré announced his cube manifolds in the short *Comptes rendus* note of 1893 which opened his series of papers on topology ([5]). This is a hint to the importance which Poincaré attributed to those manifolds.

Here is Poincaré's description of the 3-torus: Cf. Figure 1. The coordinates of the points $A, B, C, D, A', B', C', D'$ are (0,0,0), (0,1,0), (1,0,0), (1,1,0), (0,0,1), (0,1,1), (1,0,1), (1,1,1) (actually this is only important in the sixth example). In Poicaré's paper there is no picture - pictures are rather rare in his work! Here is one: see Figure 2.

With the help of the relations defined by the identification scheme - the generators of the fundamental group of the torus are the three translations - Poincaré could calculate the fundamental group. In modern notation he arrived at the result that it is isomorphic to $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. Because this fundamental group is abelian, it was easy

*Atlas page :http://www.map.him.uni-bonn.de/

Keywords: cube manifold, history of topology

Accepted: February 12th 2013

$$(1)\quad \begin{cases} ABDC \equiv A'B'D'C', \\ ACC'A' \equiv BDD'B', \\ CDD'C' \equiv ABB'A'. \end{cases}$$

Voici ce que j'entends par cette notation : la relation

$$ABDC \equiv A'B'D'C'$$

signifie :

1° Que les faces ABDC et A′B′D′C′ sont conjuguées ;

2° Que les sommets se rencontrant sur la première de ces faces se rencontrent dans l'ordre circulaire ABDC ;

3° Que les sommets A et A′, B et B′, D et D′, C et C′ se correspondent.

FIGURE 1. Poincaré's description of the 3-torus ([8, p.231])

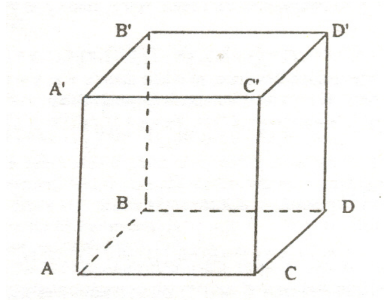

FIGURE 2. Identifications of the cube

to compute the first Betti number as 4 and by duality also the second (equal to 4 also). It should be noted that Poincaré's Betti numbers are modern Betti numbers plus 1.

The second example given by Poincaré is one in which the identifications don't provide a manifold - so it is negative in a sense. The third example is more interesting: Cf. Figure 3.

$$ABDC \equiv B'D'C'A'$$
$$ABB'A' \equiv C'CDD'$$
$$ACC'A' \equiv DD'B'B$$

FIGURE 3. Poincaré's third example

Here the translations are combined with rotations of 90°. There are two cycles of vertices: $A \equiv B' \equiv C' \equiv D$ and $B \equiv D' \equiv A' \equiv C$ and four cycles of edges: Cf. Figure 4.

The calculation of the fundamental group shows that this is isomorphic to the group of units of the quaterions. Therefore the 3-manifold of example 3 was later called (by Seifert and Threlfall ([10, p. 198]; cf. also [12]) "the quaternionian space". Poincaré himself related the group he had calculated to the symetries of the hypercube in 4-space. Calculating the Betti-numbers Poincaré found 1 and (by duality) 1. But there is a torsion coefficient in dimension one which wasn't noticed by Poincaré

A B ≡ B'D' ≡ C'C
A A' ≡ C'D' ≡ D B
A C ≡ DD' ≡ B'A'
C D ≡ B B' ≡ A'C'

FIGURE 4. Edge cycles in Poincaré's third example

in his paper of 1895; this mistake was corrected in the first complement dating from 1900 ([7], [8, 292]).

The identification scheme of Poincaré's fourth example is the following: cf. Figure 5:

A B D C ≡ B'D'C'A'
A B B'A' ≡ C D D'C'
A C C'A' ≡ B D D' B'

FIGURE 5. Poincaré's fourth example

There is only one cycle of vertices, that is, all vertices are identified but there are three cycles of edges: Cf. Figure 6:

A A' ≡ C C' ≡ B B' ≡ D D'
A B ≡ C D ≡ B'D' ≡ A'C'
A C ≡ B D ≡ D'C' ≡ B'A'

FIGURE 6. Edge cycles in Poincaré's fourth example

Poincaré did not arrive at an explicit representation of the fundamental group of example four. He calculated only the Betti numbers as 2 not taking into account torsion.

The fifth example in Poincaré's paper is derived from an octahedron, it yields the projective space.

The sixth example is a whole family of cube manifolds; its role in Poincaré's paper is central because he shows with this construction that there are 3-manifolds with same Betti numbers but different fundamental groups. So the fundamental group is a stronger invariant then the Betti numbers.

The family of 3-manifolds which Poincaré called the sixth example is defined by three 'substitutions' (that is mappings).

$$f_1 : (x, y, z) \mapsto (x + 1, y, z)$$

ABC≡ A'D'C'
BCD≡ B'A'D'
ACC'A'≡ BDD'B'
ABB'A'≡ CDD'C'

FIGURE 7. The identification scheme for the special case of Poincaré's sixth example

$$f_2 : (x, y, z) \mapsto (x, y + 1, z)$$
$$f_3 : (x, y, z) \mapsto (\alpha \cdot x + \beta \cdot y, \gamma \cdot x + \delta \cdot y, z + 1)$$

The numbers α, β, γ and δ are integers such that $\alpha \cdot \delta$ - $\beta \cdot \gamma$ equals 1. In modern language this means that the matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is an element of $SL(2, \mathbb{Z})$. We denote this matrix by $M(\alpha, \beta, \gamma, \delta)$. In contrast to the manner in which the other examples were constructed - labeled by Poincaré as 'geometric' - the sixth example is defined by the action of a group that acts discontinuously and without fixed points - Poincaré called this "Representation by a discontinuous group" ([8, 236]). In modern language the sixth example is an orbit space. From a geometric point of view there are two translations (f_1 and f_2) as in the case of the 3-torus (identiying the left face with the right face, the face in front with that in behind) plus a combination of a translation and a a composition of transvections which determines the identifiaction of the bottom with the top.

Poincaré gave some concrete examples. If we choose the matrix as the identity matrix we get the torus, if we choose it to describe a rotation about 90° we get the fourth example. Poincaré himself added the following example: If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ equals $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ then the identification scheme is as follows: Cf. Figure 7.

The fundamental group of the manifold one gets - we call it $V(\alpha, \beta, \gamma, \delta)$ - is isomorphic to the group which is generated by the three defining transformations. By a long and complicated calculation Poincaré was able to calculate the first Betti number of the sixth example. His results was the following:

If $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $\beta_1(V(\alpha, \beta, \gamma, \delta)) = 4$; if $\alpha + \delta = 2$ but the matrix $M(\alpha, \beta, \gamma, \delta)$ is not the identity matrix, then $\beta_1(M(\alpha, \beta, \gamma, \delta)) = 3$. In all other cases $\beta_1(M(\alpha, \beta, \gamma, \delta)) = 2$.

In order to get the result he was looking for - two manifolds with the same first and second Betti numbers (always by duality) but different fundamental groups - Poincaré had to study the fundamental groups of the manifolds $V(\alpha, \beta, \gamma, \delta)$. For two such matrices by M and M', the fundamental groups, G and G', of the corresponding manifolds are isomorphic to the groups of deck transformations generated by the three "substitutions" f_1, f_2 and f_3: this is because the covering is simply connected. Poincaré arrived at the following result: The groups G and G' are isomorphic if and only if the matrices M and M' are in the same conjugacy class in $SL(2,\mathbb{Z})$. So Poincaré's problem is solved: An example is given by the matrices

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & h' \\ 0 & 1 \end{pmatrix},$$

if the two numbers h and h' have different absolute value. As a concrete example we may choose the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and so on.

Remark 0.1. [9, 29] corrected Poincaré's result: The conjugacy classes must be taken in $GL(2,\mathbb{Z})$ and the matrices and their inverses have to be taken into account. But the concrete conclusion of Poincaré in not harmed by this correction.

In his dissertation (1931) Seifert came back to Poincaré's sixth example in a different context. Imagine the two identifications by f_1 and f_2 are made. Then the result is the part of 3-space between two concentric tori. The identification by f_3 has to glue the inner torus to the outer. Such a situation was called a "Schalenraum" (shell space) by Seifert ([11]). For more details see [13, 127] (historical information), [1] (group theory), [2] (Poincaré's examples in another context) and [9] (more on Poincaré's sixth example).

References

[1] B. Evans and L. Moser, *Solvable fundamental groups of compact* 3*-manifolds*, Trans. Amer. Math. Soc. **168** (1972), 189-210. MR 0301742 Zbl 0241.57002

[2] J. Hempel, 3*-Manifolds*, Princeton University Press, Princeton, N. J., 1976. MR 0415619 Zbl 1058.57001

[3] W. Killing, *Einführung in die Grundlagen der Geometrie. I. Band*, Paderborn. F. Schöningh. 1893. Zbl 25.0853.01

[4] H. Poincaré, *Memoir on Kleinian groups (Mémoire sur les groupes Kleinéens)*, Act. Math. **3** (1883) 49-92. MR 1554613 Zbl 15.0348.02

[5] H. Poincaré, *Sur l'analysis situs*, C. R. **115** (1893) 633-636. Zbl 24.0506.02

[6] H. Poincaré, *Analysis situs*, J. de l'Éc. Pol. (2) I (1895) 1-123. Zentralblatt|26.0541.07

[7] H. Poincaré, *Second complément à l'analysis situs*, J. Lond. M. S. Proc. **32** (1900) 277-308. Zbl 31.0477.10

[8] H. Poincaré, *Oeuvres. Tome VI*, Paris 1953. MR 1401792 Zbl 894.01020

[9] K. S. Sarkaria, *A look back at Poincaré's "Analysis situs"*, in: Henri Poincaré: science et philosophie (Nancy, 1994), Akademie Verlag (1996), 251–258. MR 1384996 Zbl 0851.01003

[10] H. Seifert and W. Threlfall, *Lehrbuch der Topologie*, Leipzig und Berlin: B. G. Teubner. VII, 1934. Zentralblatt|0009.08601

[11] H. Seifert, *Konstruktion dreidimensionaler geschlossener Räume*, Ber. Verh. Sächs. Akad. Leipzig **83** (1931) 26-66. Zbl 0002.16001

[12] W. Threlfall and H. Seifert, *Topologische Untersuchung der Diskontinuitätsbereiche endlicher Bewegungsgruppen des dreidimensionalen sphärischen Raumes*, Math. Ann. **104** (1931), no.1, 1–70. MR 1512649 Zbl 0006.03403

[13] K. Volkert, *Das Homöomorphieproblem, insbesondere der 3-Mannigfaltigkeiten in der Topologie 1932 - 1935*, Paris, (2002).

PROF. DR. KLAUS VOLKERT
DIDAKTIK DER MATHEMATIK
UNIVERSITÄT WUPPERTAL
GAUSSSTR. 20, RAUM F 12.06
D-42119 WUPPERTAL

E-mail address: klaus.volkert@math.uni-wuppertal.de
Web address: http://www2.math.uni-wuppertal.de/~volkert/