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Fake lens spaces*

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Abstract. A fake lens space is an orbit space of a free action of a finite cyclic group on a sphere and as such it is a generalization of a classical lens space. The invariants of fake lens spaces described here are their homotopy groups, homology groups, a certain k-invariant, the Reidemeister torsion, the ρ-invariant and certain splitting invariants. We survey the classification of fake lens spaces which includes the classification up to homotopy, up to simple homotopy and up to homeomorphism, employing methods of homotopy theory, algebraic K-theory and surgery theory. Finally we discuss the join construction which builds fake lens spaces from other fake lens spaces of a lower dimension.

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1. Introduction

A fake lens space is the orbit space of a free action of a finite cyclic group on a sphere. As such it is a topological manifold. If the action is required to be smooth then a smooth fake lens space is obtained. On this page mostly topological fake lens spaces are discussed, since for these the classification is better understood.

Clearly, classical lens spaces, which are orbit spaces of free actions on a sphere coming from unitary representations of a finite cyclic group, are examples of fake lens spaces. In order to obtain fake lens spaces which are not homeomorphic to these classical lens spaces more sophisticated technology is needed. One can either use surgery theoretic methods, or one can define certain actions of finite cyclic groups on Brieskorn varieties. See the constructions and examples in Section 7.

The classification of topological fake lens spaces can be seen as one of the basic questions in the topology of manifolds. It is systematically obtained in three stages: the homotopy classification using classical homotopy theory, the simple homotopy classification using Reidemeister torsion, and finally, surgery theory is employed to obtain a homeomorphism classification within the respective simple homotopy types. The classification of lens spaces with fundamental groups of order N with N odd and $N = 2$, was one of the first spectacular applications of surgery theory (see [3], [2], and the full classification in this setting in [14, chapters 14D, 14E]).

2. Definition

Definition 2.1. Let G be a finite cyclic group and let α be a free action of G by homeomorphisms on the sphere S^{2d-1}. A **fake lens space** is the orbit space of α and it is denoted by $L^{2d-1}(\alpha)$.

*Atlas page: www.map.mpim-bonn.mpg.de/Fake_lens_spaces

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Recall that the classical lens spaces were denoted by symbols like $L(N, l_1, \ldots, l_d)$, where N is the order of the group G. For brevity the notation $L^{2d-1}(\alpha_k) = L(N, k, 1, \ldots, 1)$ is sometimes used below. Also, when the dimension and the action are clear, we sometimes leave them from notation and simply write L.

Note that by the Lefschetz fixed point theorem, only the group of order 2 can act freely on a sphere S^{2k} of even dimension. For this case see the article on fake real projective spaces.

3. Notation

Throughout this page G will be the finite cyclic group of order N. It will have a preferred generator T which allows us to identify

$$\mathbb{Z}G = \mathbb{Z}[T]/\langle T^N - 1\rangle.$$

The norm element is $Z = 1 + T + \cdots + T^{N-1}$. Further we denote $R_G = \mathbb{Z}G/\langle Z\rangle = \mathbb{Z}[T]/\langle 1 + T + \cdots + T^{N-1}\rangle$. The projection map $\mathbb{Z}G \to R_G$ fits into the arithmetic square:

$$\begin{array}{ccc} \mathbb{Z}G & \xrightarrow{\eta} & R_G \\ \downarrow{\scriptstyle\varepsilon} & & \downarrow{\scriptstyle\varepsilon'} \\ \mathbb{Z} & \xrightarrow[\eta']{} & \mathbb{Z}_N \end{array}$$

where ε, ε' are the augmentation maps. The augmentation ideal is the kernel $I_G = \ker \varepsilon$.

We will also need the ring $\mathbb{Q}R_G = \mathbb{Q} \otimes R_G$ which we identify as

$$\mathbb{Q}R_G = \mathbb{Q}[T]/\langle 1 + T + \cdots + T^{N-1}\rangle.$$

The Pontrjagin dual of G, is the group $\widehat{G} = \mathrm{Hom}_{\mathbb{Z}}(G, S^1)$. Recall that since G is a finite cyclic group the representation ring $R(G)$ can be canonically identified with the group ring $\mathbb{Z}\widehat{G}$. Then we also have $\mathbb{Q}R(G) = \mathbb{Q} \otimes R(G) = \mathbb{Q}\widehat{G}$. Dividing out the regular representation corresponds to dividing out the norm element, hence $R(G)/\langle \mathrm{reg}\rangle = R_{\widehat{G}} = \mathbb{Z}\widehat{G}/\langle Z\rangle$ and $\mathbb{Q}R(G)/\langle \mathrm{reg}\rangle = \mathbb{Q}R_{\widehat{G}} = \mathbb{Q}\widehat{G}/\langle Z\rangle$. We will also choose a preferred generator χ of $\widehat{G}$ which gives the identifications

$$\mathbb{Q}R_{\widehat{G}} = \mathbb{Q}[\chi]/\langle 1 + \chi + \cdots + \chi^{N-1}\rangle.$$

Also note that one can vary the group G. Suppose that we have a subgroup $H \subset G$. Then by restricting the action we obtain from a fake lens space associated with the group G a fake lens space associated with the group H. Similarly we could start with a fake complex projective space which is an orbit space of a free tame action of S^1 on S^{2d-1} and restrict the action to $G \subset S^1$, to obtain a fake lens space associated with G. Both of these operations are referred to as **transfer**.

4. Invariants

For $L = L^{2d-1}(\alpha)$ we have the following invariants:

- $\pi_1(L) = \mathbb{Z}_N$, $\pi_i(L) = \pi_i(S^{2d-1})$ for $i \geq 2$,

- $H_0(L) = \mathbb{Z}$, $H_{2d-1}(L) = \mathbb{Z}$, $H_{2i-1}(L) = \mathbb{Z}_N$ for $1 \leq i \leq d-1$, $H_i(L) = 0$ for all other values of i,
- the k-invariant (in the sense of homotopy theory) $k_{2d-1}(L) \in H^{2d}(BG;\mathbb{Z})$,
- the Reidemeister torsion $\Delta(L) \in \mathbb{Q}R_G = \mathbb{Q}[T]/\langle 1 + T + \cdots T^{N-1}\rangle$ and
- the ρ-invariant $\rho(L) \in \mathbb{Q}R_{\widehat{G}}^{(-1)^d} \subset \mathbb{Q}[\chi]/\langle 1 + \chi + \cdots \chi^{N-1}\rangle$.

When $N = 2^K \cdot M$ with $\gcd(M, 2) = 1$, then we have for a manifold structure $h \colon L \longrightarrow L(\alpha_k)$ representing an element in the surgery structure set $\mathcal{S}^s(L(\alpha_k))$ the so-called **splitting invariants**:

- $\mathbf{s}_{4i}(h) \in \mathbb{Z}_{2^K}$ for $i \geq 1$,
- $\mathbf{s}_{4i-2}(h) \in \mathbb{Z}_2$ for $i \geq 1$.

The invariants $\mathbf{s}_{4i-2}(h)$ are obtained by passing to the associated manifold structure on the real projective space $\mathbb{R}P^{2d-1}$ (alias restricting the action to $\mathbb{Z}_2 \subset G$) and taking the splitting invariant along $\mathbb{R}P^{4i-2} \subset \mathbb{R}P^{2d-1}$.

The splitting invariants $\mathbf{s}_{4i}(h)$ are harder to describe. One way is as follows. It follows from the calculations of the normal invariants $\mathcal{N}(L(\alpha))$ that for d even the manifold structure h is normally cobordant to a manifold structure which comes from a manifold structure on the complex projective space $\mathbb{C}P^{d-1}$ by the transfer (alias restricting the action to $G \subset S^1$). When d is odd the same is true for the suspension manifold structure $\Sigma(h)$, defined in Section 8. The invariant $\mathbf{s}_{4i}(h)$ is then obtained from the splitting invariant along $\mathbb{C}P^{2i} \subset \mathbb{C}P^{d-1}$ which is an integer, by taking its class modulo 2^K when d is even, and from the splitting invariant along $\mathbb{C}P^{2i} \subset \mathbb{C}P^{d}$ which is also an integer, by taking its class modulo 2^K, when d is odd.

For the splitting invariants of the manifold structures on complex projective spaces see the page Fake complex projective spaces

5. Simple homotopy theory

5.1. **Preliminaries.** The homotopy classification is stated in the a priori broader context of finite CW-complexes L with $\pi_1(L) \cong \mathbb{Z}_N$ and with the universal cover homotopy equivalent to S^{2d-1} of which fake lens spaces are obviously a special case. It is convenient to make the following definition.

Definition 5.1. Let L be a CW-complex with $\pi_1(L) \cong \mathbb{Z}_N$ and with universal cover homotopy equivalent to S^{2d-1}. A **polarization** of L is a pair (T, e) where T is a choice of a generator of $\pi_1(L)$ and e is a choice of a homotopy equivalence $e \colon \widetilde{L} \to S^{2d-1}$.

Recall the classical lens space $L^{2d-1}(N; k, 1, \ldots, 1)$. By $L^i(N; k, 1, \ldots, 1)$ is denoted its i-skeleton with respect to the standard cell decomposition. If i is odd this is a lens space, if i is even this is a CW-complex obtained by attaching an i-cell to the lens space of dimension $i - 1$.

Theorem 5.2. [14, Theorem 14E.3, first part] *Let L be a finite CW-complex with $\pi_1(L) \cong \mathbb{Z}_N$ and universal cover S^{2d-1} polarized by (T, e). Then there exists a map $\phi \colon S^{2d-2} \to L^{2d-2}(N, 1, \ldots, 1)$ and a simple homotopy equivalence*

$$h \colon L \to L^{2d-2}(N; 1, \ldots, 1) \cup_\phi e^{2d-1}$$

preserving the polarization, such that the $\mathbb{Z}G$-chain complex differential on the right hand side is given by $\partial_{2d-1}e^{2d-1} = e^{2d-2}(T-1)U$ for some $U \in \mathbb{Z}G$ which maps to a unit $u \in R_G$. Furthermore, L is a simple Poincare complex and its Reidemeister torsion is $\Delta(L) = (T-1)^d \cdot u$. The element u is unique up to powers of T.

5.2. **Homotopy classification.**

Theorem 5.3. [14, Theorem 14E.3, second part] *The polarized homotopy types of such L are in one-to-one correspondence with the units in $\mathbb{Z}_N$. The correspondence is given by $L \mapsto \varepsilon'(u) \in \mathbb{Z}_N$. The invariant $\varepsilon'(u)$ can be identified with the first non-trivial k-invariant of L (in the sense of homotopy theory) $k_{2d-1}(L) \in H^{2d}(BG;\mathbb{Z})$.*

5.3. **Simple homotopy classification.**

Theorem 5.4. [14, Theorem 14E.3, third part] *The polarized simple homotopy types of such L are in one-to-one correspondence with the equivalence classes of units in R_G, where the equivalence relation is by the powers of T. The correspondence is given by $L \mapsto u \in R_G$.*

The existence of a fake lens space in the homotopy type of such L is addressed in [14, Theorem 14E.4]. Unless both N and d are even there always exists a manifold homotopy equivalent to the complex L.

5.4. **Fake lens spaces versus classical lens spaces.** Since the units $\varepsilon'(u) \in \mathbb{Z}_N$ are exhausted by the lens spaces $L^{2d-1}(N,k,1,\dots,1)$ we obtain the following corollary.

Corollary 5.5. *For any fake lens space $L^{2d-1}(\alpha)$ there exists $k \in \mathbb{N}$ and a homotopy equivalence*

$$h\colon L^{2d-1}(\alpha) \to L^{2d-1}(N;k,1,\dots,1).$$

We note that the simple homotopy classification of fake lens spaces of course includes the simple homotopy classification of the classical lens spaces. For the classical lens spaces this also already yields the homeomorphism classification, the details can be found on the page about lens spaces or in [4].

6. Homeomorphism classification

The homeomorphism classification, as already noted, is an excellent application of the non-simply connected surgery theory. Recall that for a topological manifold X the surgery theoretic homeomorphism classification of manifolds wihin the homotopy type of X is stated in terms of the simple surgery structure set $\mathcal{S}^s(X)$ and that the primary tool for its calculation is the surgery exact sequence.

The homeomorphism classification described here of course specializes to the homeomorphism classification of the classical lens spaces as can be found on the page on lens spaces, and see also the Remark 6.5 at the end of this section.

6.1. **N is odd.** In terms of the structure set the main result of [14, section 14E] can be expressed as follows.

Theorem 6.1. [14, Theorem 14E.7] *If N is odd, then the reduced ρ-invariant map*

$$\widetilde{\rho}\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \to \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$$

given by $\widetilde{\rho}(h : L \to L(\alpha)) \mapsto \rho(L) - \rho(L(\alpha))$ is injective.

However, in this case, that means for N odd, Wall managed to obtain an even better result, namely the complete classification of fake lens spaces of a given dimension $(2d-1) \geq 5$ with the fundamental group $G \cong \mathbb{Z}_N$ which is stated in the two theorems below.

The classification theorem is:

Theorem 6.2. [14, Theorem 14E.7, first part] *Let L^{2d-1} and L'^{2d-1} be oriented fake lens spaces with fundamental group G cyclic of odd order N. Then there is an orientation preserving homeomorphism $L \to L'$ inducing the identity on G if and only if $\Delta(L) = \Delta(L')$ and $\rho(L) = \rho(L')$.*

The realization theorem is:

Theorem 6.3. [14, Theorem 14E.7, second part] *Let G be cyclic of odd order. Given $\Delta \in R_G$ and $\rho \in \mathbb{Q}R_{\widehat{G}}$, there exists a corresponding fake lens space L^{2d-1} if and only if the following four statements hold:*

- *Δ and ρ are both real (d even) or imaginary (d odd).*
- *Δ generates I_G^n, $\rho \in I_{\widehat{G}}^{-n}$.*
- *The classes of $\rho \mod I_{\widehat{G}}^{-n+1}$ and $(-2)^n\Delta \mod I_G^{n+1}$ correspond under*

$$I_{\widehat{G}}^{-n}/I_{\widehat{G}}^{-n+1} \cong \widehat{H}^{2n}(\widehat{G};\mathbb{Z}) \cong \widehat{H}^{-2n}(G;\mathbb{Z}) \cong I_G^n/I_G^{n+1}.$$

- $\rho \equiv -\sum_{\phi\in\widehat{G},\phi\neq 1} \mathit{sign}(i^n\phi(\Delta))\phi \mod 4.$

6.2. **N is general.** The remaining cases were addressed in [10] and [9] from where the following theorem, stated in tems of the structure set is taken.

Theorem 6.4. [9, Theorem 1.2] *Let $L^{2d-1}(\alpha)$ be a fake lens space with $\pi_1(L^{2d-1}(\alpha)) \cong \mathbb{Z}_N$ where $N = 2^K \cdot M$ with $K \geq 0$, M odd and $d \geq 3$. Then we have*

$$(\widetilde{\rho}, \mathbf{r}_{4i-2}, \mathbf{r}_{4i})\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \cong \bar{\Sigma}_N(d) \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,1\}}} \oplus \bigoplus_{i=1}^{c} \mathbb{Z}_{2^{\min\{K,2i\}}}$$

where $\bar{\Sigma}_N(d) \subset \mathbb{Q}R_{\widehat{G}}^{(-1)^d}$ is a free abelian group. If N is odd then its rank is $(N-1)/2$. If N is even then its rank is $N/2-1$ if $d = 2e+1$ and $N/2$ if $d = 2e$. In the torsion summand we have $c = \lfloor (d-1)/2 \rfloor$.

The invariant $\widetilde{\rho}$ in Theorem 6.4 is the same reduced ρ-invariant as above. The invariants $\mathbf{r}_{4i-2} : \mathcal{S}^s(L^{2d-1}(\alpha)) \longrightarrow \mathbb{Z}_{2^{\min\{K,1\}}}$ are equal to the splitting invariants $\mathbf{s}_{4i-2}$ described in Section 4. The invariants $\mathbf{r}_{4i} : \mathcal{S}^s(L^{2d-1}(\alpha)) \longrightarrow \mathbb{Z}_{2^{\min\{K,2i\}}}$ are related to the the splitting invariants $\mathbf{s}_{4i}$ described in Section 4, but they do not have a straightforward description, only an inductive one, see [9, section 7] for more details.

6.3. **Some ideas from the proofs.** As mentioned above the strategy in all cases is to investigate the surgery exact sequence for L. In this case there is enough information about the normal invariants, the L-groups and the surgery obstruction so that one is left with just an extension problem. Briefly speaking the normal invariants can be calculated separately when localized at 2, in which case a reduction to ordinary cohomology with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$ is obtained, and away from 2, in which case a real reduced K-theory is obtained. This is well-known for $L(\alpha)$. The L-groups are completely described by the representation theory of G. By these calculations the surgery obstruction map can only be non-trivial in one case which is investigated in [14, Theorem 14E.4]. To proceed further it is convenient to study the relation of the surgery exact sequence to representation theory of G. This is done via the following commutative diagram of abelian groups and homomorphisms with exact rows.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{L}^s_{2d}(G) & \xrightarrow{\partial} & \mathcal{S}^s(L^{2d-1}(\alpha)) & \xrightarrow{\eta} & \widetilde{\mathcal{N}}(L^{2d-1}(\alpha)) & \longrightarrow & 0 \\
 & & \cong \downarrow G-sign & & \downarrow \widetilde{\rho} & & \downarrow [\widetilde{\rho}] & & \\
0 & \longrightarrow & 4 \cdot R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}} & \longrightarrow & \mathbb{Q}R^{(-1)^d}_{\widehat{G}}/4 \cdot R^{(-1)^d}_{\widehat{G}} & \longrightarrow & 0
\end{array}$$

Here the symbol $\widetilde{L}^s_{2d}(G)$ denotes the reduced L-group and the vertical isomorphism from it is the G-signature which can be found for example in [7]. The symbol $\widetilde{\mathcal{N}}(L^{2d-1}(\alpha))$ denotes the kernel of the surgery obstruction map. As mentioned above it defers from the normal invariants $\mathcal{N}(L^{2d-1}(\alpha))$ only in one case, namely when both N and d are even, and in that case the difference turns out to be a factor of $\mathbb{Z}_2$. The symbol $[\widetilde{\rho}]$ is the homomorphism induced by $\widetilde{\rho}$.

To obtain Theorem 6.1 it is shown in [14] that the map $[\widetilde{\rho}]$ is injective [14, Proposition 14E.6]. The proof proceeds by induction with respect to the dimension $2d-1$. The crucial ingredient is the transfer which produces fake lens spaces from fake complex projective spaces and the formula for the ρ-invariant of fake complex projective spaces [14, Theorem 14C.4] which by naturality of the ρ-invariant with respect to taking subgroups translates to a formula for the ρ-invariant of the associated fake lens spaces [14, Theorem 14E.8 (c)]. The induction is possible thanks to the suspension construction described in Section 8 below and the multiplicativity of the ρ-invariant with respect to this construction [14, Theorem 14A.1].

To obtain Theorem 6.4 again the map $[\widetilde{\rho}]$ is studied, first in case $N = 2^K$ when it turns out not to be injective. The kernel of $[\widetilde{\rho}]$ is determined in general in [10]. The result is again obtained via fake complex projective spaces and the formula [14, Theorem 14C.4]. However, in this case the calculation is obtained via induction on K (where remember $N = 2^K$) for any fixed $2d-1$.

Finally, in [9] the two special cases are combined to obtain the general cases via mostly formal arguments.

Remark 6.5. The homeomorphism classification of classical lens spaces due to [6] and [5], given on the lens space page, is a special case of Theorem 6.2. The additional sign and factor of k appearing in the statement of the classical theorem arise since Theorem 6.2 classifies oriented lens spaces with a fixed identification of the fundamental group. The sign allows for a change of orientation and the factor of k for a change of generator for the fundamental group.

7. Construction and examples

Classical lens spaces are of course examples of fake lens spaces. To get fake lens spaces which are not homeomorphic to classical ones one can employ the construction of fake complex projective spaces. Note that a fake complex projective space is an orbit space of a free tame action of S^1 on S^{2d-1} and that we obviously have $\mathbb{Z}_N \cong G < S^1$. Restricting the action to the subgroup we obtain a fake lens space. Its ρ-rho-invariant can be calculated by naturality using the formula for the ρ-invariant of the circle action.

The above construction does not exhaust all the fake lens spaces. To get all of them there is a construction which produces from a given fake lens space L another fake lens space L' such that the difference of their ρ-invariants is a prescribed element

$$\rho(L) - \rho(L') = x \in 4 \cdot R_{\widehat{G}}^{(-1)^d}.$$

The construction is just the Wall realization from surgery theory, alias a non-simply connected generalization of the plumbing construction.

Another possibilty is to obtain fake lens spaces as orbit spaces of actions of G on Brieskorn varieties. This was pursued for example in [11].

8. The join construction / The suspension map

Let G be a group acting freely on the spheres S^m and S^n. Then the two actions extend to the join $S^{m+n+1} \cong S^m * S^n$ and the resulting action remains free.

Given two fake lens spaces L and L', one can pass to the universal covers, form the join and then pass to the quotient again. The resulting space is again a fake lens space. This operation is called the join and denoted by $L * L'$, or $L(\alpha * \alpha') = L(\alpha) * L(\alpha')$.

Given two manifold structures $h\colon L \xrightarrow{\simeq_s} L(\alpha)$ and $h\colon L' \xrightarrow{\simeq_s} L(\alpha')$, one can pass to the induced maps of the universal covers, extend them to a map of the joins and pass to the map of quotients. This will again be a simple homotopy equivalence and hence a manifold structure $h * h'\colon L * L' \xrightarrow{\simeq_s} L(\alpha * \alpha')$

When $L' = L^1(N,1) = L^1(\alpha_1)$ this operation is called a suspension. Taking $h' = \mathrm{id}$ in the above paragraph defines a map

$$\Sigma\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \to \mathcal{S}^s(L^{2d+1}(\alpha * \alpha_1)).$$

Theorem 8.1. [14, Corollary on page 228 in section 14E] *If N is odd and $d \geq 3$ then the map*

$$\Sigma\colon \mathcal{S}^s(L^{2d-1}(\alpha)) \to \mathcal{S}^s(L^{2d+1}(\alpha * \alpha_1))$$

is bijective.

The proof is based on the classification theorem above.

Theorem 8.2. [9, Theorem 6.1 and 6.2] *For N even and $e \geq 1$ there are exact sequences*

$$0 \to \mathcal{S}^s(L_N^{4e+1}(\alpha_k)) \xrightarrow{\Sigma} \mathcal{S}^s(L_N^{4e+3}(\alpha_k)) \xrightarrow{\sigma} \mathbb{Z} \to 0,$$

$$0 \to \mathbb{Z} \to \mathcal{S}^s(L^{4e+3}_N(\alpha_k)) \xrightarrow{\Sigma} \mathcal{S}^s(L^{4e+5}_N(\alpha_k)) \xrightarrow{\mu} \mathbb{Z}_2 \to 0.$$

The invariants σ and μ (alias desuspension obstructions) are obtained by passing to the associated fake real projective spaces via the transfer (alias restricting the group action to $\mathbb{Z}_2$) and taking the Browder-Livesay invariants described in [8] and [14, Chapter 12]. The invariant σ can be identified with the ρ-invariant associated to manifolds with $\pi_1 = \mathbb{Z}_2$ (in which case it is just an integer).

The proof is based on the classification theorem above and also on the proofs of the analogous theorems for $N = 2$ described on the page fake real projective spaces.

9. Further sources

The sources mentioned in the text are those where the final classification statements were presented. They built on the previous work, some of which is contained in the following papers: [1], [3], [12], [13], [2].

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