Bulletin of the Manifold Atlas (2011) 49–55

5-manifolds: 1-connected*

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Abstract. We recall the classification of closed smooth simply-connected 5-manifolds begun by Smale in 1962 and completed by Barden in 1965. We also give a construction of every such manifold which differs from Barden's original construction.

57R99, 57R65

1. Introduction

Let $\mathcal{M}_5$ be the set of diffeomorphism classes of closed, oriented, smooth, simply-connected 5-manifolds M and let $\mathcal{M}_5^{\mathrm{Spin}} \subset \mathcal{M}_5$ be the subset of diffeomorphism classes of spinable manifolds. The calculation of $\mathcal{M}_5^{\mathrm{Spin}}$ was first obtained by Smale [9] and was one of the first applications of the h-cobordism theorem. A little latter Barden [1] devised an elegant surgery argument and applied results of [15] on the diffeomorphism groups of 4-manifolds to give an explicit and complete classification of all of $\mathcal{M}_5$.

Simply-connected 5-manifolds are an appealing class of manifolds: the dimension is just large enough so that the full power of surgery techniques can be applied but it is low enough that the manifolds are simple enough to be readily classified. A feature of simply-connected 5-manifolds is that the homotopy, homeomorphism and diffeomorphism classification all coincide. Note that not every simply-connected 5-dimensional Poincaré space is smoothable. The classification of simply-connected 5-dimensional Poincaré spaces was achieved by Stöcker [11].

2. Constructions and examples

We first list some familiar 5-manifolds using Barden's notation:

- $X_0 := S^5$.
- $M_\infty := S^2 \times S^3$.
- $X_\infty := S^2 \tilde{\times}_\gamma S^3$, the total space of the non-trivial S^3-bundle over S^2.
- $X_{-1} := SU_3/SO_3$, the Wu-manifold, is the homogeneous space obtained from the standard inclusion of $SO_3 \to SU_3$.

In [1, Section 1] a construction of simply-connected 5-manifolds is given by expressing them as twisted doubles $M \cong W \cup_f W$ where W is a certain simply connected 5-manifold with boundary ∂W a simply-connected 4-manifold and $f : \partial W \cong \partial W$ is a diffeomorphism. Barden used results of [15] to show that diffeomorphisms realising the required ismorphisms of $H_2(\partial W)$ exist.

*Atlas page: http://www.map.mpim-bonn.mpg.de/5-manifolds:_1-connected

Keywords: smooth manifold, complex manifold

Accepted: 22nd December 2010

2.1. **The general spin case.** Next we present a construction of simply-connected spin 5-manifolds. Note that all homology groups are with integer coefficients. Given a finitely generated abelian group G, let X_G denote the degree 2 Moore space with $H_2(X_G) = G$. The space X_G may be realised as a finite CW-complex with only 2-cells and 3-cells and so there is an embedding $X_G \to \mathbb{R}^6$. Let N_G be a regular neighbourhood of $X_G \subset \mathbb{R}^6$ and let M_G be the boundary of N_G. Then M_G is a closed, smooth, simply-connected, spinable 5-manifold with $H_2(M_G) \cong G \oplus TG$ where TG is the torsion subgroup of G. For example, $M_{\mathbb{Z}^r} \cong \sharp_r(S^2 \times S^3)$ where $\sharp_r$ denotes the r-fold connected sum.

2.2. **The general non-spin case.** For the non-spin case we construct only those manifolds which are boundaries of 6-manifolds. Let (G, w) be a pair with $w\colon G \to \mathbb{Z}_2$ a surjective homomorphism and G as above. We shall construct a non-spin 5-manifold $M_{(G,w)}$ with $H_2(M_{(G,w)}) \cong G \oplus TG$ and second Stiefel-Whitney class w_2 given by w composed with the projection $G \oplus TG \to G$.

If $(G, w) = (\mathbb{Z}, 1)$ let $N_{(\mathbb{Z},1)}$ be the non-trivial D^4-bundle over S^2 with boundary $\partial N_{(\mathbb{Z},1)} = M_{(\mathbb{Z},1)} = X_\infty$. If $(G, w) = (\mathbb{Z}, 1) \oplus (\mathbb{Z}^r, 0)$ let $N_{(G,w)}$ be the boundary connected sum $N_{(\mathbb{Z},1)}\natural_r(S^2 \times D^4)$ with boundary $M_{(G,w)} = X_\infty\sharp_r(S^2 \times S^3)$.

In the general case, present $G = F/i(R)$ where $i : R \to F$ is an injective homomorphism between free abelian groups. Lift (G, w) to $(F, \bar{w})$ and observe that there is a canonical identification $F = H_2(M_{(F,\bar{w})})$. If $\{r_1, \dots, r_n\}$ is a basis for R note that each $i(r_i) \in H_2(M_{(F,\bar{w})})$ is represented by a an embedded 2-sphere with trivial normal bundle. Let $N_{(G,w)}$ be the manifold obtained by attaching 3-handles to $N_{(F,\bar{w})}$ along spheres representing $i(r_i)$ and let $M_{(G,w)} = \partial N_{(G,w)}$. One may check that $M_{(G,w)}$ is a non-spin manifold as described above.

3. INVARIANTS

Consider the following invariants of a closed simply-connected 5-manifold M.

- $H_2(M)$ be the second integral homology group of M, with torsion subgroup $TH_2(M)$.
- $w_2\colon H_2(M) \to \mathbb{Z}_2$, the homomorphism defined by evaluation with the second Stiefel-Whitney class of M, $w_2 \in H^2(M; \mathbb{Z}_2)$.
- $h(M) \in \mathbb{N} \cup \{\infty\}$, the smallest extended natural number r such that $x^{2^r} = e$ and $x \in w_2^{-1}(1)$. If M is spin we set $h(M) = 0$.
- $b_M : TH_2(M) \times TH_2(M) \to \mathbb{Q}/\mathbb{Z}$, the linking form of M which is a non-singular anti-symmetric bi-linear pairing on $TH_2(M)$.

By [13, Proposition 1 & 2] the linking form satisfies the identity $b_M(x, x) = w_2(x)$ where we regard $w_2(x)$ as an element of $\{0, 1/2\} \subset \mathbb{Q}/\mathbb{Z}$.

For example, the Wu-manifold X_{-1} has $H_2(X_{-1}) = \mathbb{Z}_2$, non-trivial w_2, $h(X_{-1}) = 1$ and linking form b_{-1} defined below.

3.1. **Linking forms.** An abstract non-singular anti-symmetric linking form on a finite abelian group H is a bi-linear function

$$b : H \times H \to \mathbb{Q}/\mathbb{Z}$$

such that $b(x, y) = 0$ for all $y \in H$ if and only if $x = 0$ and $b(x, y) = -b(y, x)$ for all pairs x and y. For example, we have the following linking forms specified by their linking matricies

$$b_{-1} : C_2 \times C_2 \to \mathbb{Q}/\mathbb{Z}, \quad \begin{pmatrix} \frac{1}{2} \end{pmatrix},$$

$$b_0(C_k) : (C_k \oplus C_k) \times (C_k \oplus C_k) \to \mathbb{Q}/\mathbb{Z}, \quad \begin{pmatrix} 0 & \frac{1}{k} \\ -\frac{1}{k} & 0 \end{pmatrix},$$

$$b_j(C_{2^j}) : (C_{2^j} \oplus C_{2^j}) \times (C_{2^j} \oplus C_{2^j}) \to \mathbb{Q}/\mathbb{Z}, \quad \begin{pmatrix} \frac{1}{2} & \frac{1}{2^j} \\ -\frac{1}{2^j} & 0 \end{pmatrix}.$$

If $T = \oplus_{i=1}^r C_{k_r}$ is the sum of cyclic groups we shall write $b_0(T)$ for the sum $\oplus_{i=1}^r b_0(C_{k_r})$.

By [14, Theorem 3] all non-singular anti-symmetric linking forms are isomorphic to a sum of the linking forms above. Indeed such linking forms are classified up to isomorphism by the homomorphism

$$w(b) : H \to \mathbb{Z}_2, \quad x \mapsto b(x, x).$$

Moreover H must be isomorphic to $T \oplus T$ or $T \oplus T \oplus \mathbb{Z}_2$ for some finite group T with $b(x, x) = 1/2$ if x generates the $\mathbb{Z}_2$ summand. In particular the second Stiefel-Whitney class of a 5-manifold M determines the isomorphism class of the linking form b_M and we see that the torsion subgroup of $H_2(M)$ is of the form $TH_2(M) \cong T \oplus T$ if $h(M) \neq 1$ or $TH_2(M) \cong T \oplus T \oplus \mathbb{Z}_2$ if $h(M) = 1$ in which case the $\mathbb{Z}_2$ summand is an orthogonal summand of b_M.

3.2. **Values for constructions.** The spin manifolds M_G all have vanishing w_2 of course and so by Wall's classification of linking forms we see that the linking form of M_G is the linking form $b_0(TG)$.

As we mentioned above, the non-spin manifolds $M_{(G,\omega)}$ have w_2 given by projecting to G and then applying ω:

$$w_2 = \omega \circ pr : TG \oplus G \to G \to \mathbb{Z}/2.$$

If $M_{(G,\omega)}$ has height finite height $h(M_{(G,\omega)}) = j$ then it follows from Wall's classification of linking forms that $b_{M_{(G,\omega)}} \cong b_1(C_{2^j}) \oplus b_0(T)$ where $TG \cong C_{2^j} \oplus T$ and if $M_{(G,\omega)}$ has infinite height then $b_{M_{(G,\omega)}} = b_0(TG)$.

4. Classification

We first present the most economical classifications of $\mathcal{M}_5^{\mathrm{Spin}}$ and $\mathcal{M}_5$. Let $\mathcal{A}b^{T\oplus T\oplus *}$ be the set of isomorphism classes finitely generated abelian groups G with torsion subgroup $TG \cong H \oplus H \oplus C$ where C is trivial or $C \cong \mathbb{Z}_2$ and write $\mathcal{A}b^{T\oplus T}$ and $\mathcal{A}b^{T\oplus T\oplus \mathbb{Z}_2}$ for the obvious subsets of $\mathcal{A}b^{T\oplus T\oplus *}$.

Theorem 4.1 (Smale [9])**.** *There is a bijective correspondence*

$$\mathcal{M}_5^{Spin} \to \mathcal{A}b^{T\oplus T}, \quad [M] \mapsto [H_2(M)].$$

Theorem 4.2 (Barden [1]). *The mapping*

$$\mathcal{M}_5 \to \mathcal{A}b^{T \oplus T \oplus *} \times (\mathbb{N} \cup \{\infty\}), \quad [M] \mapsto ([H_2(M)], h(M))$$

is an injection onto the subset of pairs $([G], n)$ *where* $[G] \in \mathcal{A}b^{T \oplus T \oplus \mathbb{Z}_2}$ *if and only if* $n = 1$.

The above theorems follow from the following theorem of Barden and the classification of anti-symmetric linking forms.

Theorem 4.3 (Barden [1], Theorem 2.2). *Let* M_0 *and* M_1 *be simply-connected, closed, smooth 5-manifolds and let* $A : H_2(M_0) \cong H_2(M_1)$ *be an isomorphism preserving the linking form and the second Stiefel-Whitney class. Then* A *is realised by a diffeomorphism* $f_A : M_0 \cong M_1$.

This theorem can re-phrased in categorical language as follows.

- Let $\mathcal{Q}_5$ be the groupoid with objects (G, b, w) where G is a finitely generated abelian group, $b\colon TG \times TG \to \mathbb{Q}/\mathbb{Z}$ is an anti-symmetric non-singular linking form and $w : G \to \mathbb{Z}_2$ is a homomorphism such that $w(x) = b(x, x)$ for all $x \in TG$. The morphisms of $\mathcal{Q}_5$ are isomorphisms of abelian groups commuting with both w and b.
- Let $\widetilde{\mathcal{M}}_5$ be the groupoid with objects simply-connected closed smooth 5-manifolds embedded in some fixed $\mathbb{R}^N$ for N large and morphisms isotopy classes of diffeomorphisms.
- Consider the functor

$$(b, w_2) : \widetilde{\mathcal{M}}_5 \to \mathcal{Q}_5 : \ M \mapsto (H_2(M), b_M, w_2(M)), \ f : M_0 \cong M_1 \mapsto H_2(f).$$

Theorem 4.4 (Barden [1]). *The functor* $(b, w_2) : \widetilde{\mathcal{M}}_5 \to \mathcal{Q}_5$ *is a detecting functor. That is, it induces a bijection on isomorphism classes of objects.*

4.1. **Enumeration.** We first give Barden's enumeration of the set $\mathcal{M}_5$, [1, Theorem 2.3].

- $X_0 := S^5$, $M_\infty := S^2 \times S^3$, $X_\infty := S^2 \tilde{\times}_\gamma S^3$, $X_{-1} := SU_3/SO_3$.
- For $1 < k < \infty$, $M_k = M_{\mathbb{Z}_k}$ is the spin manifold with $H_2(M) = \mathbb{Z}_k \oplus \mathbb{Z}_k$ constructed in Section 2.1 above.
- For $1 < j < \infty$ let $X_j = M_{(\mathbb{Z}_{2^j}, 1)}$ be the non-spin manifold with $H_2(X_j) \cong \mathbb{Z}_{2^j} \oplus \mathbb{Z}_{2^j}$ construction in Section 2.2 above.

With this notation [1, Theorem 2.3] states that a complete list of diffeomorphism classes of simply-connected 5-manifolds is given by

$$X_{j,k_1,\dots,k_n} = X_j \sharp M_{k_1} \sharp \dots \sharp M_{k_n}$$

where $-1 \leq j \leq \infty$, $1 < k_i$, k_i divides k_{i+1} or $k_{i+1} = \infty$ and $\sharp$ denotes the connected sum of oriented manifolds. The manifold $X_{j',k'_1,\dots k'_n}$ is diffeomorphic to $X_{j,k_1,\dots,k_n}$ if and only if $(j', k'_1, \dots, k'_n) = (j, k_1, \dots, k_n)$.

An alternative complete enumeration is obtained by writing $\mathcal{M}_5$ as a disjoint union

$$\mathcal{M}_5 = \mathcal{M}_5^{\text{Spin}} \sqcup \mathcal{M}_5^{w_2,=\partial} \sqcup \mathcal{M}_5^{w_2,\neq\partial}$$

where the last two sets denote the diffeomorphism classes of non-spinable 5-manifolds which are respectively boundaries and not boundaries. Then

$$\mathcal{M}_5^{\mathrm{Spin}} = \{[M_G]\},\ \mathcal{M}_5^{w_2,=\partial} = \{[M_{(G,\omega)}]\} \text{ and } \mathcal{M}_5^{w_2,\neq\partial} = \{[X_{-1}\sharp M_G]\}.$$

5. Further discussion

- As the invariants which classify simply-connected closed oriented 5-manifolds are homotopy invariants, we see that the same classification holds up to homotopy, homeomorphism and piecewise linear homeomorphism.
- By the construction in Section 2.1 above every simply-connected closed smooth spinable 5-manifold embedds into $\mathbb{R}^6$.
- As the invariants for $-M$ are isomorphic to the invariants of M we see that every smooth 5-manifold admits an orientation reversing diffeomorphism: i.e. all 5-manifolds are smoothly amphicheiral.
- Barden's results have been nicely discussed and re-proven by Zhubr [16].

5.1. **Bordism groups.** As $\mathcal{M}_5^{\mathrm{Spin}} = \{[M_G]\}$, $M_G = \partial N_G$ and M_G admits a unique spin structure which extends to N_G we see that every closed spin 5-manifold bounds a spin 6-manifold. Hence the bordism group Ω_5^{Spin} vanishes.

The bordism group Ω_5^{SO} is isomorphic to $\mathbb{Z}_2$, see for example [8, p 203]. Moreover this bordism group is detected by the Stiefel-Whitney number $\langle w_2(M)w_3(M), [M]\rangle \in \mathbb{Z}_2$. The Wu-manifold has cohomology groups

$$H^*(X_{-1};\mathbb{Z}_2) = \mathbb{Z}_2, 0, \mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}_2, \quad * = 0,1,2,3,4,5,$$

and $w_2(X_{-1}) \neq 0$. It follows that $w_3(X_{-1}) = \mathrm{Sq}^1(w_2(X_{-1})) \neq 0$ and so we have that $\langle w_2(X_{-1})w_3(X_{-1}), [X_{-1}]\rangle \neq 0$. We see that $[X_{-1}]$ is the generator of Ω_5^{SO} and that a closed, smooth 5-manifold M is not a boundary if and only if it is diffeomorphic to $X_{-1}\sharp M_0$ where M_0 is a Spin manifold.

5.2. **Curvature and contact structures.** Every manifold $\sharp_r(S^2 \times S^3)$ admits a metric of positive Ricci curvature [3]. Every simply connected 5-manifold admits a contact structure [6]. The special case of this latter theorem for spin 5-manifolds with the order of $TH_2(M)$ prime to 3 was proven in [12].

5.3. **Mapping class groups.** Let $\pi_0\,\mathrm{Diff}_+(M)$ denote the group of isotopy classes of orientation preserving diffeomorphisms $f : M \cong M$ and let $\mathrm{Aut}(H_2(M))$ be the group of isomorphisms of $H_2(M)$ preserving the linking form and the second Stiefel-Whitney class. Applying Theorem 4.3 above we obtain the following exact sequence

$$0 \to \pi_0\,\mathrm{SDiff}(M) \to \pi_0\mathrm{Diff}_+(M) \to \mathrm{Aut}(H_2(M)) \to 0 \quad (*)$$

where $\pi_0\,\mathrm{SDiff}(M)$ is the group of isotopy classes of diffeomorphisms inducing the identity on $H_*(M)$.

- There is an isomphorism $\pi_0\,\mathrm{Diff}_+(S^5) \cong 0$. By [4] and [10], $\pi_0\,\mathrm{Diff}_+(S^5) \cong \Theta_6$, the group of homotopy 6-spheres. But by [7], $\Theta_6 \cong 0$.
- In the homotopy category, $\mathcal{E}_+(M)$, the group of homotopy classes of orientation preserving homotopy equivalences of M, has been extensively investigated by [2] and is already seen to be relatively complex.

- Open problem: as of writing there is no computation of $\pi_0 \operatorname{SDiff}(M)$ for a general simply-connected 5-manifold in the literature.
  (1) However if $TH_2(M)$ has no 2-torsion and no 3-torsion then $\pi_0 \operatorname{SDiff}(M)$ was computed in [5]. This computation agrees with a more recent conjectured answer: please see the discussion page of the corresponding Manifold Atlas page.
  (2) Even the computation of $\pi_0 \operatorname{SDiff}(M)$ still leaves an unsolved extension problem in $(*)$ above.

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