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Complex bordism*

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Abstract. We give the basic definitions for the complex bordism groups of manifolds and survey some foundational results in the subject.

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1. Introduction

Complex bordism (also known as *unitary bordism*) is the bordism theory of stably complex manifolds (see Section 2). It is one of the most important theories of bordism with additional structure, or B-bordism.

The theory of complex bordism is much richer than its unoriented analogue, and at the same time is not as complicated as oriented bordism or other bordism theories with additional structure (B-bordism). Thanks to this, complex cobordism theory has found the most striking and important applications in algebraic topology and beyond. Many of these applications, including the formal group techniques and the Adams-Novikov spectral sequence (see Section 7) were outlined in the pioneering work [10].

2. Stably complex structures

A direct attempt to define the bordism relation on complex manifolds fails because the manifold W is odd-dimensional and therefore cannot be complex. In order to work with complex manifolds in the bordism theory, one needs to weaken the notion of a complex structure. This leads directly to considering *stably complex* (also known as *weakly almost complex*, *stably almost complex* or *quasicomplex*) manifolds.

Let $\mathcal{T}M$ denote the tangent bundle of M, and $\underline{\mathbb{R}}^k$ the product vector bundle $M \times \mathbb{R}^k$ over M. A *tangential stably complex structure* on M is determined by a choice of an isomorphism

$$c_{\mathcal{T}} \colon \mathcal{T}M \oplus \underline{\mathbb{R}}^k \to \xi$$

between the 'stable' tangent bundle and a complex vector bundle ξ over M. Some of the choices of such isomorphisms are deemed to be equivalent, i.e. determine the same stably complex structures (see details in Chapters II and VII of [14]). In particular, two stably complex structures are equivalent if they differ by a trivial complex summand. A *normal stably complex structure* on M is determined by a choice of a complex bundle structure on the normal bundle $\nu(M)$ of an embedding $M \hookrightarrow \mathbb{R}^N$. Tangential and normal stably complex structures on M determine each

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other by means of the canonical isomorphism $\mathcal{T}M \oplus \nu(M) \cong \underline{\mathbb{R}}^N$. We therefore may restrict our attention to tangential structures only.

A *stably complex manifold* is a pair $(M, c_\mathcal{T})$ consisting of a manifold M and a stably complex structure $c_\mathcal{T}$ on it. This is a generalisation of a complex and *almost complex* manifold (where the latter means a manifold with a choice of a complex structure on $\mathcal{T}M$, i.e. a stably complex structure $c_\mathcal{T}$ with $k = 0$).

Example 2.1. Let $M = \mathbb{C}P^1$. The standard complex structure on M is equivalent to the stably complex structure determined by the isomorphism

$$\mathcal{T}(\mathbb{C}P^1) \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \overline{\eta} \oplus \overline{\eta}$$

where η is the Hopf line bundle. On the other hand, the isomorphism

$$\mathcal{T}(\mathbb{C}P^1) \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \eta \oplus \overline{\eta} \cong \underline{\mathbb{C}}^2$$

determines a trivial stably complex structure on $\mathbb{C}P^1$.

3. Definition of bordism and cobordism

The bordism relation can be defined between stably complex manifolds. Like the case of unoriented bordism, the set of bordism classes $[M, c_\mathcal{T}]$ of stably complex manifolds of dimension n is an Abelian group with respect to the disjoint union. This group is called the *n-dimensional complex bordism group* and denoted $\varOmega_n^U$. The zero element is represented by the bordism class of any manifold M which bounds and whose stable tangent bundle is trivial (and therefore isomorphic to a product complex vector bundle $M \times \mathbb{C}^k$). The sphere S^n provides an example of such a manifold. The opposite element to the bordism class $[M, c_\mathcal{T}]$ in the group $\varOmega_n^U$ may be represented by the same manifold M with the stably complex structure determined by the isomorphism

$$\mathcal{T}M \oplus \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^2 \xrightarrow{c_\mathcal{T} \oplus e} \xi \oplus \underline{\mathbb{C}}$$

where $e \colon \mathbb{R}^2 \to \mathbb{C}$ is given by $e(x, y) = x - iy$.

An abbreviated notation $[M]$ for the complex bordism class will be used whenever the stably complex structure $c_\mathcal{T}$ is clear from the context.

The *complex bordism group* $U_n(X)$ and *cobordism group* $U^n(X)$ of a space X may also be defined geometrically, at least for the case when X is a manifold. This can be done along the lines suggested by [11] and [6] by considering special 'stably complex' maps of manifolds M to X. However, nowadays the homotopical approach to bordism has taken over, and the (co)bordism groups are usually defined using the Pontrjagin-Thom construction similarly to the unoriented case:

$$U_n(X) = \lim_{k \to \infty} \pi_{2k+n}((X_+) \wedge MU(k)),$$
$$U^n(X) = \lim_{k \to \infty} [\Sigma^{2k-n}(X_+), MU(k)]$$

where $MU(k)$ is the Thom space of the universal complex k-plane bundle $EU(k) \to BU(k)$, and $[X, Y]$ denotes the set of homotopy classes of pointed maps from X to Y. These groups are $\varOmega_*^U$-modules and give rise to a multiplicative (co)homology theory. In particular, $U^*(X) = \oplus_n U^n(X)$ is a graded ring.

The graded ring Ω_U^* with $\Omega_U^n = \Omega_{-n}^U$ is called the *complex cobordism ring*; it has nontrivial elements only in nonpositively graded components.

4. Geometric cobordisms

There is one important case when certain cobordism classes can be represented very explicitly by maps of manifolds.

For any cell complex X the cohomology group $H^2(X)$ can be identified with the set $[X, \mathbb{C}P^\infty]$ of homotopy classes of maps into $\mathbb{C}P^\infty$. Since $\mathbb{C}P^\infty = MU(1)$, every element $x \in H^2(X)$ also determines a cobordism class $u_x \in U^2(X)$. The elements of $U^2(X)$ obtained in this way are called *geometric cobordisms* of X. We therefore may view $H^2(X)$ as a subset in $U^2(X)$, however the group operation in $H^2(X)$ is not obtained by restricting the group operation in $U^2(X)$ (see Formal group laws and genera for the relationship between the two operations).

When X is a manifold, geometric cobordisms may be described by submanifolds $M \subset X$ of codimension 2 with a fixed complex structure on the normal bundle.

Indeed, every $x \in H^2(X)$ corresponds to a homotopy class of maps $f_x \colon X \to \mathbb{C}P^\infty$. The image $f_x(X)$ is contained in some $\mathbb{C}P^N \subset \mathbb{C}P^\infty$, and we may assume that $f_x(X)$ is transverse to a certain hyperplane $H \subset \mathbb{C}P^N$. Then $M_x := f_x^{-1}(H)$ is a codimension 2 submanifold in X whose normal bundle acquires a complex structure by restriction of the complex structure on the normal bundle of $H \subset \mathbb{C}P^N$. Changing the map f_x within its homotopy class does not affect the bordism class of the embedding $M_x \to X$.

Conversely, assume given a submanifold $M \subset X$ of codimension 2 whose normal bundle is endowed with a complex structure. Then the composition

$$X \to M(\nu) \to MU(1) = \mathbb{C}P^\infty$$

of the Pontrjagin-Thom collapse map $X \to M(\nu)$ and the map of Thom spaces corresponding to the classifying map $M \to BU(1)$ of ν defines an element $x_M \in H^2(X)$, and therefore a geometric cobordism.

If X is an oriented manifold, then a choice of complex structure on the normal bundle of a codimension 2 embedding $M \subset X$ is equivalent to orienting M. The image of the fundamental class of M in the homology of X is Poincaré dual to $x_M \in H^2(X)$.

5. Structure results

The complex bordism ring Ω_*^U is described as follows.

Theorem 5.1. (1) *$\Omega_*^U \otimes \mathbb{Q}$ is a polynomial ring over $\mathbb{Q}$ generated by the bordism classes of complex projective spaces $\mathbb{C}P^i$, $i \geqslant 1$.*
(2) *Two stably complex manifolds are bordant if and only if they have identical sets of Chern characteristic numbers.*
(3) *Ω_*^U is a polynomial ring over $\mathbb{Z}$ with one generator a_i in every even dimension $2i$, where $i \geqslant 1$.*

Part 1 can be proved by the methods of [15]. Part 2 follows from the results of [7] and [8]. Part 3 is the most difficult one; it was done by [8] using the Adams spectral sequence and structure theory of Hopf algebras (see also [9] for a more detailed account) and Milnor (unpublished, but see [16]) in 1960. Another more geometric proof was given by [13], see also [14].

6. Multiplicative generators

6.1. **Preliminaries: characteristic numbers detecting generators.** To describe a set of multiplicative generators for the ring Ω^U_* we shall need a special characteristic class of complex vector bundles. Let ξ be a complex k-plane bundle over a manifold M. Write its total Chern class formally as follows:

$$c(\xi) = 1 + c_1(\xi) + \cdots + c_k(\xi) = (1 + x_1) \cdots (1 + x_k),$$

so that $c_i(\xi) = \sigma_i(x_1, \ldots, x_k)$ is the ith elementary symmetric function in formal indeterminates. These indeterminates acquire a geometric meaning if ξ is a sum $\xi_1 \oplus \cdots \oplus \xi_k$ of line bundles; then $x_j = c_1(\xi_j)$, $1 \leqslant j \leqslant k$. Consider the polynomial

$$P_n(x_1, \ldots x_k) = x_1^n + \cdots + x_k^n$$

and express it via the elementary symmetric functions:

$$P_n(x_1, \ldots, x_k) = s_n(\sigma_1, \ldots, \sigma_k).$$

Substituting the Chern classes for the elementary symmetric functions we obtain a certain characteristic class of ξ:

$$s_n(\xi) = s_n(c_1(\xi), \ldots, c_k(\xi)) \in H^{2n}(M).$$

This characteristic class plays an important role in detecting the polynomial generators of the complex bordism ring, because of the following properties (which follow immediately from the definition).

Proposition 6.1. (1) *$s_n(\xi) = 0$ for $2n > \dim M$.*
(2) *$s_n(\xi \oplus \eta) = s_n(\xi) + s_n(\eta)$.*

Given a stably complex manifold $(M, c_{\mathcal{T}})$ of dimension $2n$, define its characteristic number

$$s_n[M] = s_n(\xi)\langle M \rangle \in \mathbb{Z}$$

where ξ is the complex bundle from the definition of stably complex structure in Section 2, and $\langle M \rangle \in H_{2n}(M)$ the fundamental homology class.

Corollary 6.2. *If a bordism class $[M] \in \Omega^U_{2n}$ decomposes as $[M_1] \times [M_2]$ where $\dim M_1 > 0$ and $\dim M_2 > 0$, then $s_n[M] = 0$.*

It follows that the characteristic number s_n vanishes on decomposable elements of Ω^U_{2n}. It also detects indecomposables that may be chosen as polynomial generators. In fact, the following result is a byproduct of the calculation of Ω^U_*:

Theorem 6.3. *A bordism class $[M] \in \Omega^U_{2n}$ may be chosen as a polynomial generator a_n of the ring Ω^U_* if and only if*

$$s_n[M] = \begin{cases} \pm 1, & \text{if } n \neq p^k - 1 \text{ for any prime } p; \\ \pm p, & \text{if } n = p^k - 1 \text{ for some prime } p. \end{cases}$$

(Ed Floyd was fond of calling the characteristic numbers $s_n[M]$ the 'magic numbers' of manifolds.)

6.2. **Milnor hypersurfaces.** A universal description of connected manifolds representing the polynomial generators $a_n \in \Omega^U_*$ is unknown. Still, there is a particularly nice family of manifolds whose bordism classes generate the whole ring Ω^U_*. This family is redundant though, so there are algebraic relations between their bordism classes.

Fix a pair of integers $j \geqslant i \geqslant 0$ and consider the product $\mathbb{C}P^i \times \mathbb{C}P^j$. Its algebraic subvariety

$$H_{ij} = \{(z_0 : \ldots : z_i) \times (w_0 : \ldots : w_j) \in \mathbb{C}P^i \times \mathbb{C}P^j \colon z_0w_0 + \cdots + z_iw_i = 0\}$$

is called a *Milnor hypersurface.* Note that $H_{0j} \cong \mathbb{C}P^{j-1}$.

The Milnor hypersurface H_{ij} may be identified with the set of pairs (l, α), where l is a line in $\mathbb{C}^{i+1}$ and α is a hyperplane in $\mathbb{C}^{j+1}$ containing l. The projection $(l, \alpha) \mapsto l$ describes H_{ij} as the total space of a bundle over $\mathbb{C}P^i$ with fibre $\mathbb{C}P^{j-1}$.

Denote by p_1 and p_2 the projections of $\mathbb{C}P^i \times \mathbb{C}P^j$ onto the first and second factors respectively, and by η the Hopf line bundle over a complex projective space; then $\bar\eta$ is the hyperplane section bundle. We have

$$H^*(\mathbb{C}P^i \times \mathbb{C}P^j) = \mathbb{Z}[x, y]/(x^{i+1} = 0,\ y^{j+1} = 0)$$

where $x = p_1^*c_1(\bar\eta)$, $y = p_2^*c_1(\bar\eta)$.

Proposition 6.4. *The geometric cobordism in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding to the element $x+y \in H^2(\mathbb{C}P^i \times \mathbb{C}P^j)$ is represented by the submanifold H_{ij}. In particular, the image of the fundamental class $\langle H_{ij}\rangle$ in $H_{2(i+j-1)}(\mathbb{C}P^i \times \mathbb{C}P^j)$ is Poincaré dual to $x + y$.*

Proof. We have $x + y = c_1(p_1^*(\bar\eta) \otimes p_2^*(\bar\eta))$. The classifying map $f_{x+y} \colon \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^\infty$ is the composition of the *Segre embedding*

$$\sigma \colon \mathbb{C}P^i \times \mathbb{C}P^j \to \mathbb{C}P^{ij+i+j},$$
$$(z_0 : \ldots : z_i) \times (w_0 : \ldots : w_j) \mapsto (z_0w_0 : z_0w_1 : \ldots : z_kw_l : \ldots : z_iw_j),$$

and the embedding $\mathbb{C}P^{ij+i+j} \to \mathbb{C}P^\infty$. The codimension 2 submanifold in $\mathbb{C}P^i \times \mathbb{C}P^j$ corresponding to the cohomology class $x+y$ is obtained as the inverse image $\sigma^{-1}(H)$ of a generally positioned hyperplane in $\mathbb{C}P^{ij+i+j}$ (i.e. a hyperplane H transverse to the image of the Segre embedding). By its definition, the Milnor hypersurface is exactly $\sigma^{-1}(H)$ for one of such hyperplanes H. □

Lemma 6.5. *We have*

$$s_{i+j-1}[H_{ij}] = \begin{cases} j, & \text{if } i = 0, \text{ i.e. } H_{ij} = \mathbb{C}P^{j-1}; \\ 2, & \text{if } i = j = 1; \\ 0, & \text{if } i = 1,\ j > 1; \\ -\binom{i+j}{i}, & \text{if } i > 1. \end{cases}$$

Proof. Let $i = 0$. Since the stably complex structure on $H_{0j} = \mathbb{C}P^{j-1}$ is determined by the isomorphism $\mathcal{T}(\mathbb{C}P^{j-1}) \oplus \mathbb{C} \cong \bar\eta \oplus \ldots \oplus \bar\eta$ (j summands) and $x = c_1(\bar\eta)$, we have

$$s_{j-1}[\mathbb{C}P^{j-1}] = jx^{j-1}\langle \mathbb{C}P^{j-1}\rangle = j.$$

Now let $i > 0$. Then

$$s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)) = (i+1)x^{i+j-1} + (j+1)y^{i+j-1} = \begin{cases} 2x^j + (j+1)y^j, & \text{if } i = 1; \\ 0, & \text{if } i > 1. \end{cases}$$

Denote by ν the normal bundle of the embedding $\iota\colon H_{ij} \to \mathbb{C}P^i \times \mathbb{C}P^j$. Then

$$\mathcal{T}(H_{ij}) \oplus \nu = \iota^*(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)).$$

Since $c_1(\nu) = \iota^*(x+y)$, we obtain $s_{i+j-1}(\nu) = \iota^*(x+y)^{i+j-1}$.

Assume $i = 1$. Then by the previous Proposition,

$$\begin{aligned} s_j[H_{1j}] = s_j(\mathcal{T}(H_{1j}))\langle H_{1j}\rangle &= \iota^*(2x^j + (j+1)y^j - (x+y)^j)\langle H_{1j}\rangle \\ &= (2x^j + (j+1)y^j - (x+y)^j)(x+y)\langle \mathbb{C}P^1 \times \mathbb{C}P^j\rangle \\ &= \begin{cases} 2, & \text{if } j = 1; \\ 0, & \text{if } j > 1. \end{cases} \end{aligned}$$

Assume now that $i > 1$. Then $s_{i+j-1}(\mathcal{T}(\mathbb{C}P^i \times \mathbb{C}P^j)) = 0$, and by the previous Proposition,

$$\begin{aligned} s_{i+j-1}[H_{ij}] = -s_{i+j-1}(\nu)\langle H_{ij}\rangle &= -\iota^*(x+y)^{i+j-1}\langle H_{ij}\rangle \\ &= -(x+y)^{i+j}\langle \mathbb{C}P^i \times \mathbb{C}P^j\rangle = -\binom{i+j}{i}, \end{aligned}$$

which finishes the proof of the Lemma. □

Theorem 6.6. *The bordism classes $\{[H_{ij}],\ 0 \leqslant i \leqslant j\}$ multiplicatively generate the complex bordism ring $\varOmega^U_*$.*

Proof. This follows from the fact that

$$\text{g.c.d.}\left(\tbinom{n+1}{i},\ 1 \leqslant i \leqslant n\right) = \begin{cases} p, & \text{if } \ n = p^k - 1, \\ 1, & \text{otherwise}, \end{cases}$$

and the previous Lemma. □

Example 6.7. We list some bordism groups and generators:

- $\varOmega^U_{2i+1} = 0$;
- $\varOmega^U_0 = \mathbb{Z}$, generated by a point;
- $\varOmega^U_2 = \mathbb{Z}$, generated by $[\mathbb{C}P^1]$, as $1 = 2^1 - 1$ and $s_1[\mathbb{C}P^1] = 2$;
- $\varOmega^U_4 = \mathbb{Z} \oplus \mathbb{Z}$, generated by $[\mathbb{C}P^1 \times \mathbb{C}P^1]$ and $[\mathbb{C}P^2]$, as $2 = 3^1 - 1$ and $s_2[\mathbb{C}P^2] = 3$;
- $[\mathbb{C}P^3]$ cannot be taken as the polynomial generator $a_3 \in \varOmega^U_6$, since $s_3[\mathbb{C}P^3] = 4$, while $s_3(a_3) = \pm 2$. The bordism class $[H_{22}] + [\mathbb{C}P^3]$ may be taken as a_3.

The previous theorem about the multiplicative generators for $\varOmega^U_*$ has the following important addendum.

Theorem 6.8 (Milnor)**.** *Every bordism class $x \in \varOmega^U_n$ with $n > 0$ contains a non-singular algebraic variety (not necessarily connected).*

(The Milnor hypersufaces are algebraic, but one also needs to represent $-[H_{ij}]$ by algebraic varieties!) For the proof see Chapter 7 of [14].

The following question is still open, even in complex dimension 2.

Problem 6.9 (Hirzebruch)**.** Describe the set of bordism classes in Ω^U_* containing connected nonsingular algebraic varieties.

Example 6.10. Every class $k[\mathbb{C}P^1] \in \Omega^U_2$ contains a nonsingular algebraic variety, namely, a disjoint union of k copies of $\mathbb{C}P^1$ for $k > 0$ and a Riemannian surface of genus $(1 - k)$ for $k \leqslant 0$. Connected algebraic varieties are only contained in the bordism classes $k[\mathbb{C}P^1]$ with $k \leqslant 1$.

6.3. **Toric generators and quasitoric representatives in cobordism classes.** There is an alternative set of multiplicative generators $\{[B_{ij}], 0 \leqslant i \leqslant j\}$ for the complex bordism ring Ω^U_*, consisting of nonsingular projective toric varieties, or *toric manifolds*. Every B_{ij} therefore supports an effective action of a 'big torus' (of dimension half the dimension of the manifold) with isolated fixed points. The construction of B_{ij} is due to [4] (see also [3] and [2]).

The Milnor hypersurfaces H_{ij} (see Section 6.2) are not toric manifolds for $i > 1$, because of a simple cohomological obstruction (see Proposition 5.43 in [3]).

The manifold B_{ij} is constructed as the projectivisation of a sum of j line bundles over the *bounded flag manifold* B_i.

A *bounded flag* in $\mathbb{C}^{n+1}$ is a complete flag

$$\mathcal{U} = \{U_1 \subset U_2 \subset \ldots \subset U_{n+1} = \mathbb{C}^{n+1},\ \ \dim U_i = i\}$$

for which U_k, $\ 2 \leqslant k \leqslant n$, contains the coordinate subspace $\mathbb{C}^{k-1}$ spanned by the first $k - 1$ standard basis vectors.

The set B_n of all bounded flags in $\mathbb{C}^{n+1}$ is a smooth complex algebraic variety of dimension n (cf. [4]), referred to as the *bounded flag manifold*. The action of the algebraic torus $(\mathbb{C}^\times)^n$ on $\mathbb{C}^{n+1}$ given by

$$(t_1, \ldots, t_n) \cdot (w_1, \ldots, w_n, w_{n+1}) = (t_1w_1, \ldots, t_nw_n, w_{n+1}),$$

where $(t_1, \ldots, t_n) \in (\mathbb{C}^\times)^n$ and $(w_1, \ldots, w_n, w_{n+1}) \in \mathbb{C}^{n+1}$, induces an action on bounded flags, and therefore endows B_n with a structure of a toric manifold.

B_n is also the total space of a *Bott tower*, that is, a tower of fibrations with base $\mathbb{C}P^1$ and fibres $\mathbb{C}P^1$ in which every stage is the projectivisation of a sum of two line bundles. In particular, B_2 is the Hirzebruch surface H_1.

The manifold B_{ij} $(0 \leqslant i \leqslant j)$ consists of pairs $(\mathcal{U}, W)$, where $\mathcal{U}$ is a bounded flag in $\mathbb{C}^{i+1}$ and W is a line in $U_1^\perp \oplus \mathbb{C}^{j-i}$. (Here $U_1^\perp$ denotes the orthogonal complement to U_1 in $\mathbb{C}^{i+1}$, so that $U_1^\perp \oplus \mathbb{C}^{j-i}$ is the orthogonal complement to U_1 in $\mathbb{C}^{j+1}$.) Therefore, B_{ij} is the total space of a bundle over B_i with fibre $\mathbb{C}P^{j-1}$. This bundle is in fact the projectivisation of a sum of j line bundles, which implies that B_{ij} is a complex $2(i + j - 1)$-dimensional toric manifold.

The bundle $B_{ij} \to B_i$ is the pullback of the bundle $H_{ij} \to \mathbb{C}P^i$ along the map $f\colon B_i \to \mathbb{C}P^i$ taking a bounded flag $\mathcal{U}$ to its first line $U_1 \subset \mathbb{C}^{i+1}$. This is described by the diagram

$$\begin{array}{ccc} B_{ij} & \longrightarrow & H_{ij} \\ \downarrow & & \downarrow \\ B_i & \xrightarrow{f} & \mathbb{C}P^i \end{array}.$$

(The bundle $H_{ij} \to \mathbb{C}P^i$, unlike $B_{ij} \to B_i$, is not a projectivisation of a sum of line bundles, which prevents the torus action on $\mathbb{C}P^i$ from lifting to an action on the total space.)

Lemma 6.11. *We have* $s_{i+j-1}[B_{ij}] = s_{i+j-1}[H_{ij}]$.

Proof. We may assume that $j > 1$, as otherwise $B_{ij} = H_{ij} = \mathbb{C}\mathrm{P}^1$. We have the equality $H_{ij} = \mathbb{C}\mathrm{P}(\xi)$, the projectivisation of a j-plane bundle ξ over $\mathbb{C}\mathrm{P}^i$. We also have that the map $f\colon B_i \to \mathbb{C}\mathrm{P}^i$ has degree 1 since it is an isomorphism on the affine chart $\{\mathcal{U} \in B_i : U_1 \not\subset \mathbb{C}^i\}$. Furthermore, $B_{ij} = \mathbb{C}\mathrm{P}(f^*\xi)$. The result now follows from Lemma 6.12 below. □

Lemma 6.12. *Let $f\colon M \to N$ be a degree d map of $2i$-dimensional almost complex manifolds, and let ξ be a complex j-plane bundle over N, $j > 1$. Then*

$$s_{i+j-1}[\mathbb{C}P(f^*\xi)] = d \cdot s_{i+j-1}[\mathbb{C}P(\xi)].$$

Proof. Let $p\colon \mathbb{C}P(\xi) \to N$ be the projection, γ the tautological bundle over $\mathbb{C}P(\xi)$, and $\gamma^\perp$ the complementary bundle, so that $\gamma \oplus \gamma^\perp = p^*(\xi)$. Then we have

$$\mathcal{T}(\mathbb{C}P(\xi)) = p^*\mathcal{T}N \oplus \mathcal{T}_F(\mathbb{C}P(\xi)),$$

where $\mathcal{T}_F(\mathbb{C}P(\xi))$ is the tangent bundle along the fibres of the projection p. Since $\mathcal{T}_F(\mathbb{C}P(\xi)) = \mathrm{Hom}(\gamma, \gamma^\perp)$ and $\mathrm{Hom}(\gamma,\gamma) = \underline{\mathbb{C}}$ (a trivial complex line bundle), we obtain

$$\mathcal{T}_F(\mathbb{C}P(\xi)) \oplus \underline{\mathbb{C}} = \mathrm{Hom}(\gamma, \gamma \oplus \gamma^\perp).$$

Therefore,

(1)
$$\mathcal{T}(\mathbb{C}P(\xi)) \oplus \underline{\mathbb{C}} = p^*\mathcal{T}N \oplus \mathrm{Hom}(\gamma, \gamma \oplus \gamma^\perp) = p^*\mathcal{T}N \oplus \mathrm{Hom}(\gamma, p^*\xi) = p^*\mathcal{T}N \oplus (\bar\gamma \otimes p^*\xi),$$

where $\bar\gamma = \mathrm{Hom}(\gamma, \underline{\mathbb{C}})$.

The map f induces the map $F\colon \mathbb{C}P(f^*\xi) \to \mathbb{C}P(\xi)$ with the following properties:

(a) $pF = fp_1$, where $p_1\colon \mathbb{C}P(f^*\xi) \to M$ is the projection;
(b) $\deg F = \deg f$;
(c) $F^*\gamma$ is the tautological bundle over $\mathbb{C}P(f^*\xi)$.

Using (1), we obtain

$$s_{i+j-1}\big(\mathcal{T}(\mathbb{C}P(\xi))\big) = p^*s_{i+j-1}(\mathcal{T}N) + s_{i+j-1}(\bar\gamma \otimes p^*\xi) = s_{i+j-1}(\bar\gamma \otimes p^*\xi)$$

(since $i + j - 1 > i$), and similarly for $\mathcal{T}(\mathbb{C}P(f^*\xi))$. Thus,

$$\begin{aligned}
s_{i+j-1}[\mathbb{C}P(f^*\xi)] &= s_{i+j-1}\Big(\mathcal{T}(\mathbb{C}P(f^*\xi))\Big)\Big\langle\mathbb{C}P(f^*\xi)\Big\rangle \\
&= s_{i+j-1}\Big((F^*\bar{\gamma}) \otimes p_1^* f^*\xi\Big)\Big\langle\mathbb{C}P(f^*\xi)\Big\rangle \\
&= s_{i+j-1}\Big(F^*(\bar{\gamma} \otimes p^*\xi)\Big)\Big\langle\mathbb{C}P(f^*\xi)\Big\rangle \\
&= s_{i+j-1}(\bar{\gamma} \otimes p^*\xi)\langle F_*\mathbb{C}P(f^*\xi)\rangle \\
&= s_{i+j-1}(\bar{\gamma} \otimes p^*\xi)\langle d \cdot \mathbb{C}P(\xi)\rangle \\
&= d \cdot s_{i+j-1}[\mathbb{C}P(\xi)].
\end{aligned}$$

□

The proof of Lemma 6.11. We note that the map $f\colon B_i \to \mathbb{C}P^i$ has degree 1. (It is an isomorphism on the affine chart $\{\mathcal{U} \in B_i\colon U_1 \not\subset \mathbb{C}^i\}$.) □

Theorem 6.13 (Buchstaber and Ray [4])**.** *The bordism classes of toric manifolds $\{[B_{ij}],\ 0 \leqslant i \leqslant j\}$ multiplicatively generate the complex bordism ring Ω^U_*. Therefore, every complex bordism class contains a disjoint union of toric manifolds.*

Proof. The first statement follows from the fact that the Milnor hypersurfaces generate the complex bordism ring and the previous Lemma. A product of toric manifolds is toric, but a disjoint union of toric manifolds is not a toric manifold, since toric manifolds are connected by definition. □

The manifolds H_{ij} and B_{ij} are not bordant in general, although $H_{0j} = B_{0j} = \mathbb{C}P^{j-1}$ and $H_{1j} = B_{1j}$ by definition.

Connected representatives in cobordism classes cannot be found within toric manifolds because of severe restrictions on their characteristic numbers. (For example, the Todd genus of every toric manifold is 1.) A topological generalisation of toric manifolds was suggested in [5] (see also [3]). These manifolds have become known as *quasitoric*. A quasitoric manifold is a smooth manifold of dimension $2n$ with a *locally standard* action of an n-dimensional torus whose quotient is a simple polytope. Quasitoric manifolds generally fail to be complex or even almost complex, but they always admit stably complex structures [4].

Theorem 6.14 (Buchstaber, Panov and Ray [2])**.** *In dimensions > 2, every complex cobordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is compatible with the action of the torus.*

7. Adams-Novikov spectral sequence

A principal motivation for [10] was to develop a version of the Adams spectral sequence in which mod p cohomology (and the Steenrod algebra) are replaced by complex cobordism theory (and its ring of stable cohomology operations), for the purpose of computing stable homotopy groups. The foundations for the Adams-Novikov spectral sequence were laid in this paper, and many applications and computations have followed. An introduction to the work of Novikov on complex cobordism is given in [1]. The most comprehensive study of the Adams-Novikov spectral sequence is [12], currently available in a second edition from AMS/Chelsea.

References

[1] J. F. Adams, *Stable homotopy and generalised homology*, University of Chicago Press, Chicago, Ill., 1974. MR0402720 Zbl0309.55016

[2] V. M. Buchstaber, T. E. Panov and N. Ray, *Spaces of polytopes and cobordism of quasitoric manifolds*, Mosc. Math. J. **7** (2007), no.2, 219-242, 350. MR2337880 Zbl1176.55004

[3] V. M. Buchstaber and T. E. Panov, *Torus actions and their applications in topology and combinatorics*, American Mathematical Society, 2002. MR1897064 Zbl1012.52021

[4] V. M. Buchstaber and N. Ray, *Tangential structures on toric manifolds, and connected sums of polytopes*, Internat. Math. Res. Notices(2001), no.4, 193-219. MR1813798 Zbl0996.52013

[5] M. W. Davis and T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no.2, 417-451. MR1104531 Zbl0733.52006

[6] A. Dold, *Geometric cobordism and the fixed point transfer*, in Algebraic topology (Proc. Conf., Univ. British Columbia, Vancouver, B.C., 1977), Lecture Notes in Math. **673**, Springer, Berlin, (1978), 32–87. MR517084 Zbl0386.57005

[7] J. Milnor, *On the cobordism ring Ω^* and a complex analogue. I*, Amer. J. Math. **82** (1960), 505-521. MR0119209 Zbl0095.16702

[8] S. P. Novikov, *Some problems in the topology of manifolds connected with the theory of Thom spaces*, Soviet Math. Dokl. **1** (1960), 717-720. MR0121815 Zbl0094.35902

[9] S. P. Novikov, *Homotopy properties of Thom complexes*, Mat. Sb. (N.S.) **57 (99)** (1962), 407-442. MR0157381 Zbl0193.51801

[10] S. P. Novikov, *Methods of algebraic topology from the point of view of cobordism theory*, Math. USSR, Izv. **1**, (1967) 827–913. MR0221509 Zbl0176.52401

[11] D. Quillen, *Elementary proofs of some results of cobordism theory using Steenrod operations*, Advances in Math. **7** (1971), 29-56 (1971). MR0290382 Zbl0214.50502

[12] D. C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Academic Press Inc., Orlando, FL, 1986. MR860042 Zbl1073.55001

[13] R. E. Stong, *Relations among characteristic numbers. I*, Topology **4** (1965), 267-281. MR0192515 Zbl0136.20503

[14] R. E. Stong, *Notes on cobordism theory*, Princeton University Press, Princeton, N.J., 1968. MR0248858 Zbl0277.57010

[15] R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17-86. MR0061823 Zbl0057.15502

[16] R. Thom, *Travaux de Milnor sur le cobordisme*, (1995), Exp. No. 180, 169-177. MR1603465 Zbl0116.40402

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