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Bordism*

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Abstract. We give the basic definitions for the bordism groups of manifolds and survey some foundational results in the subject.

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1. Introduction

The theory of bordism is one of the most deep and influential parts of algebraic topology. The foundations of bordism were laid in the pioneering works of Pontrjagin [6] and Thom [8], and the theory experienced a spectacular development in the 1960s. In particular, Atiyah [1] showed that bordism is a generalised homology theory and related it to the emergent K-theory. The main introductory reference is the monograph [7].

Basic geometric constructions of bordism and cobordism, as well as homotopical definitions are summarised here. For more information, see the pages in the Bordism category of the Manifold Atlas.

2. The bordism relation

All manifolds here are assumed to be smooth, compact and closed (without boundary), unless otherwise specified. Given two n-dimensional manifolds M_1 and M_2, a *bordism* between them is an $(n+1)$-dimensional manifold W with boundary, whose boundary is the disjoint union of M_1 and M_2, that is, $\partial W = M_1 \sqcup M_2$. If such a W exists, M_1 and M_2 are called *bordant*. The bordism relation splits manifolds into equivalence classes (see Figure 1), which are called *bordism classes*.

3. Unoriented bordism

We denote the bordism class of M by $[M]$, and denote by $\varOmega_n^O$ the set of bordism classes of n-dimensional manifolds. Then $\varOmega_n^O$ is an abelian group with respect to the disjoint union operation: $[M_1] + [M_2] = [M_1 \sqcup M_2]$. Zero is represented by the bordism class of an empty set (which is counted as a manifold in any dimension), or by the bordism class of any manifold which bounds. We also have $\partial(M \times I) = M \sqcup M$. Hence, $2[M] = 0$ and $\varOmega_n^O$ is a 2-torsion group.

Set $\varOmega_*^O := \bigoplus_{n\geq 0} \varOmega_n^O$. The product of bordism classes, namely $[M_1] \times [M_2] = [M_1 \times M_2]$, makes $\varOmega_*^O$ a graded commutative ring known as the *unoriented bordism ring*.

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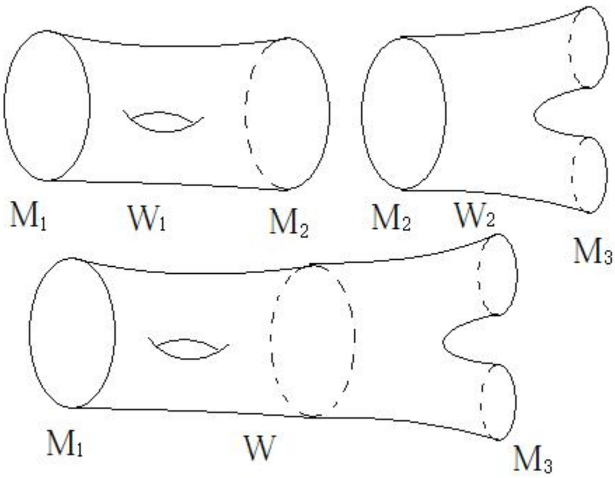

Figure 1. Transitivity of the bordism relation

For any space X the bordism relation can be extended to maps of n-dimensional manifolds to X: two maps $M_1 \to X$ and $M_2 \to X$ are *bordant* if there is a bordism W between M_1 and M_2 and the map $M_1 \sqcup M_2 \to X$ extends to a map $W \to X$. The set of bordism classes of maps $M \to X$ forms an abelian group called the *n-dimensional unoriented bordism group of X* and denoted $O_n(X)$ (other notations: $\mathfrak{N}_n(X)$, $MO_n(X)$).

The assignment $X \mapsto O_*(X)$ defines a generalised homology theory, that is, it is functorial in X, homotopy invariant, has the excision property and exact sequences of pairs. For this theory we have $O_*(pt) = \varOmega_*^O$, and $O_*(X)$ is an $\varOmega_*^O$-module.

The Pontrjagin-Thom construction reduces the calculation of the bordism groups to a homotopical problem:

$$O_n(X) = \lim_{k\to\infty} \pi_{k+n}\Big((X_+) \wedge MO(k)\Big)$$

where $X_+ = X \sqcup pt$, and $MO(k)$ is the Thom space of the universal vector k-plane bundle $EO(k) \to BO(k)$. The *cobordism groups* are defined dually:

$$O^n(X) = \lim_{k\to\infty} [\Sigma^{k-n}(X_+), MO(k)]$$

where $[X, Y]$ denotes the set of based homotopy classes of maps from X to Y. The resulting generalised cohomology theory is multiplicative, which implies that $O^*(X) = \bigoplus_n O^n(X)$ is a graded commutative ring. It follows from the definitions that $O^n(pt) = O_{-n}(pt)$. The graded ring $\varOmega_O^*$ with $\varOmega_O^{-n} := O^{-n}(pt) = \varOmega_n^O$ is called the *unoriented cobordism ring*. It has nonzero elements only in nonpositively graded components. The bordism ring $\varOmega_*^O$ and the cobordism ring $\varOmega_O^*$ differ only by their gradings, so the notions of the 'bordism class' and 'cobordism class' of a manifold M are interchangeable. The difference between bordism and cobordism appears only when one considers generalised homology and cohomology theories.

4. ORIENTED AND COMPLEX BORDISM

The bordism relation may be extended to manifolds endowed with some additional structure, which leads to the most important examples of bordism theories. The universal homotopical framework for geometric bordism with additional structure is provided by the theory of B-bordism.

The simplest additional structure is an orientation. By definition, two oriented n-dimensional manifolds M_1 and M_2 are *oriented bordant* if there is an oriented $(n+1)$-dimensional manifold W with boundary such that $\partial W = M_1 \sqcup \overline{M}_2$, where $\overline{M}_2$ denotes M_2 with the orientation reversed. The *oriented bordism groups* $\varOmega_n^{SO}$ and the *oriented bordism ring* $\varOmega_*^{SO} = \bigoplus_{n\ge 0} \varOmega_n^{SO}$ are defined accordingly. Given an oriented manifold M, the manifold $M \times I$ has a canonical orientation such that $\partial(M \times I) = M \sqcup \overline{M}$. Hence, $-[M] = [\overline{M}]$ in $\varOmega_n^{SO}$. Unlike $\varOmega_n^{O}$, elements of $\varOmega_*^{SO}$ generally do not have order 2.

Complex structure gives another important example of an additional structure on manifolds. However, a direct attempt to define the bordism relation on complex manifolds fails because the manifold W is odd-dimensional and therefore cannot be complex. This can be remedied by considering *stably complex* (also known as *weakly almost complex*, *stably almost complex* or *quasicomplex*) structures.

Let $\mathcal{T}M$ denote the tangent bundle of M, and $\underline{\mathbb{R}}^k$ the product vector bundle $M \times \mathbb{R}^k$ over M. A *tangential stably complex structure* on M is determined by a choice of an isomorphism

$$c_{\mathcal{T}} \colon \mathcal{T}M \oplus \underline{\mathbb{R}}^k \to \xi$$

between the 'stable' tangent bundle and a complex vector bundle ξ over M. Some of the choices of such isomorphisms are deemed to be equivalent, i.e. determine the same stably complex structures (see details in Chapters II and VII of [7]). In particular, two stably complex structures are equivalent if they differ by a trivial complex summand. A *normal stably complex structure* on M is determined by a choice of a complex bundle structure on the normal bundle $\nu(M)$ of an embedding $M \hookrightarrow \mathbb{R}^N$. Tangential and normal stably complex structures on M determine each other by means of the canonical isomorphism $\mathcal{T}M \oplus \nu(M) \cong \underline{\mathbb{R}}^N$. We therefore may restrict our attention to tangential structures only.

A *stably complex manifold* is a pair $(M, c_{\mathcal{T}})$ consisting of a manifold M and a stably complex structure $c_{\mathcal{T}}$ on it. This is a generalisation of a complex and *almost complex* manifold (where the latter means a manifold with a choice of a complex structure on $\mathcal{T}M$, i.e. a stably complex structure $c_{\mathcal{T}}$ with $k = 0$).

Example 4.1. Let $M = \mathbb{C}P^1$. The standard complex structure on M is equivalent to the stably complex structure determined by the isomorphism

$$\mathcal{T}(\mathbb{C}P^1) \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \overline{\eta} \oplus \overline{\eta}$$

where η is the Hopf line bundle. On the other hand, the isomorphism

$$\mathcal{T}(\mathbb{C}P^1) \oplus \underline{\mathbb{R}}^2 \xrightarrow{\cong} \eta \oplus \overline{\eta} \cong \underline{\mathbb{C}}^2$$

determines a trivial stably complex structure on $\mathbb{C}P^1$.

The bordism relation can be defined between stably complex manifolds. Like the case of unoriented bordism, the set of bordism classes $[M, c_{\mathcal{T}}]$ of n-dimensional stably complex manifolds is an Abelian group with respect to the disjoint union. This group is called the *n-dimensional complex bordism group* and denoted by Ω_n^U. The zero is represented by the bordism class of any manifold M which bounds and whose stable tangent bundle is trivial (and therefore isomorphic to a product complex vector bundle $M \times \mathbb{C}^k$). The sphere S^n provides an example of such a manifold. The opposite element to the bordism class $[M, c_{\mathcal{T}}]$ in the group Ω_n^U may be represented by the same manifold M with the stably complex structure determined by the isomorphism

$$\mathcal{T}M \oplus \underline{\mathbb{R}}^k \oplus \underline{\mathbb{R}}^2 \xrightarrow{c_{\mathcal{T}} \oplus e} \xi \oplus \underline{\mathbb{C}}$$

where $e\colon \mathbb{R}^2 \to \mathbb{C}$ is given by $e(x,y) = x - iy$.

An abbreviated notation $[M]$ for the complex bordism class will be used whenever the stably complex structure $c_{\mathcal{T}}$ is clear from the context.

The *complex bordism and cobordism groups* of a space X are defined similarly to the unoriented case:

$$\begin{aligned} U_n(X) &= \lim_{k\to\infty} \pi_{2k+n}((X_+) \wedge MU(k)), \\ U^n(X) &= \lim_{k\to\infty} [\Sigma^{2k-n}(X_+), MU(k)] \end{aligned}$$

where $MU(k)$ is the Thom space of the universal complex k-plane bundle $EU(k) \to BU(k)$. These groups are Ω_*^U-modules and give rise to a multiplicative (co)homology theory. In particular, $U^*(X) = \bigoplus_n U^n(X)$ is a graded ring.

The graded ring Ω_U^* with $\Omega_U^n = \Omega_{-n}^U$ is called the *complex cobordism ring*; it has nontrivial elements only in nonpositively graded components.

5. CONNECTED SUM AND BORDISM

For manifolds of positive dimension the disjoint union $M_1 \sqcup M_2$ representing the sum of bordism classes $[M_1] + [M_2]$ may be replaced by their 'connected sum', which represents the same bordism class.

The connected sum $M_1 \# M_2$ of manifolds M_1 and M_2 of the same dimension n is constructed as follows. Choose points $v_1 \in M_1$ and $v_2 \in M_2$, and take closed ε-balls $B_\varepsilon(v_1)$ and $B_\varepsilon(v_2)$ around them (both manifolds may be assumed to be endowed with a Riemannian metric). Fix an isometric embedding f of a pair of standard ε-balls $D^n \times S^0$ (here $S^0 = \{0, 1\}$) into $M_1 \sqcup M_2$ which maps $D^n \times 0$ onto $B_\varepsilon(v_1)$ and $D^n \times 1$ onto $B_\varepsilon(v_2)$. If both M_1 and M_2 are oriented we additionally require the embedding f to preserve the orientation on the first ball and reverse in on the second. Now, using this embedding, replace in $M_1 \sqcup M_2$ the pair of balls $D^n \times S^0$ by a 'pipe' $S^{n-1} \times D^1$. After smoothing the angles in the standard way we obtain a smooth manifold $M_1 \# M_2$.

If both M_1 and M_2 are connected the smooth structure on $M_1 \# M_2$ does not depend on a choice of points v_1, v_2 and embedding $D^n \times S^0 \hookrightarrow M_1 \sqcup M_2$. It does however depend on the orientations; $M_1 \# M_2$ and $M_1 \# \overline{M_2}$ are not diffeomorphic in general.

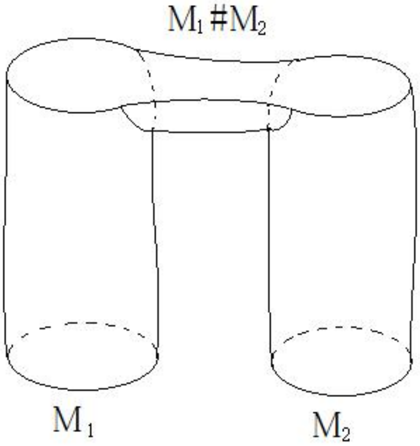

FIGURE 2. Disjoint union and connected sum

There are smooth contraction maps $p_1\colon M_1 \# M_2 \to M_1$ and $p_2\colon M_1 \# M_2 \to M_2$. In the oriented case the manifold $M_1 \# M_2$ can be oriented in such a way that both contraction maps preserve the orientations.

A bordism between $M_1 \sqcup M_2$ and $M_1 \# M_2$ may be constructed as follows. Consider a cylinder $M_1 \times I$, from which we remove an ε-neighbourhood $U_\varepsilon(v_1 \times 1)$ of the point $v_1 \times 1$. Similarly, remove the neighbourhood $U_\varepsilon(v_2 \times 1)$ from $M_2 \times I$ (each of these two neighbourhoods can be identified with the half of a standard open $(n+1)$-ball). Now connect the two remainders of cylinders by a 'half pipe' $S^n_{\le} \times I$ in such a way that the half-sphere $S^n_{\le} \times 0$ is identified with the half-sphere on the boundary of $U_\varepsilon(v_1 \times 1)$, and $S^n_{\le} \times 1$ is identified with the half-sphere on the boundary of $U_\varepsilon(v_2 \times 1)$. Smoothening the angles we obtain a manifold with boundary $M_1 \sqcup M_2 \sqcup (M_1 \# M_2)$ (or $\overline{M_1} \sqcup \overline{M_2} \sqcup (M_1 \# M_2)$ in the oriented case), see the Figure.

If M_1 and M_2 are stably complex manifolds, then there is a canonical stably complex structure on $M_1 \# M_2$, which is constructed as follows. Assume the stably complex structures on M_1 and M_2 are determined by isomorphisms

$$c_{\mathcal{T},1}\colon \mathcal{T}M_1 \oplus \underline{\mathbb{R}}^{k_1} \to \xi_1 \quad \text{and} \quad c_{\mathcal{T},2}\colon \mathcal{T}M_2 \oplus \underline{\mathbb{R}}^{k_2} \to \xi_2.$$

Using the isomorphism $\mathcal{T}(M_1 \# M_2) \oplus \underline{\mathbb{R}}^n \cong p_1^*\mathcal{T}M_1 \oplus p_2^*\mathcal{T}M_2$, we define a stably complex structure on $M_1 \# M_2$ by the isomorphism

$$\mathcal{T}(M_1 \# M_2) \oplus \underline{\mathbb{R}}^{n+k_1+k_2} \cong p_1^*\mathcal{T}M_1 \oplus \underline{\mathbb{R}}^{k_1} \oplus p_2^*\mathcal{T}M_2 \oplus \underline{\mathbb{R}}^{k_2} \xrightarrow{c_{\mathcal{T},1} \oplus c_{\mathcal{T},2}} p_1^*\xi_1 \oplus p_2^*\xi_2.$$

This stably complex structure is called the *connected sum of stably complex structures* on M_1 and M_2. The corresponding complex bordism class is $[M_1] + [M_2]$.

6. STRUCTURE RESULTS

The theory of unoriented (co)bordism was first to be completed: its coefficient ring Ω^O_* was calculated by Thom, and the bordism groups $O_*(X)$ of cell complexes X were reduced to homology groups of X with coefficients in Ω^O_*. The corresponding results are summarised as follows:

Theorem 6.1. (1) *Two manifolds are unoriented bordant if and only if they have identical sets of Stiefel-Whitney characteristic numbers.*

(2) $\varOmega_*^O$ *is a polynomial ring over* $\mathbb{Z}/2$ *with one generator* a_i *in every positive dimension* $i \neq 2^k - 1$.

(3) *For every cell complex* X *the module* $O_*(X)$ *is a free graded* $\varOmega_*^O$*-module isomorphic to* $H_*(X; \mathbb{Z}/2) \otimes_{\mathbb{Z}/2} \varOmega_*^O$.

Parts 1 and 2 were proved in [8]. Part 3 was proved in [2].

Calculating the complex bordism ring $\varOmega_*^U$ turned out to be a much more difficult problem:

Theorem 6.2. (1) $\varOmega_*^U \otimes \mathbb{Q}$ *is a polynomial ring over* $\mathbb{Q}$ *generated by the bordism classes of complex projective spaces* $\mathbb{C}P^i$, $i \geq 1$.

(2) *Two stably complex manifolds are bordant if and only if they have identical sets of Chern characteristic numbers.*

(3) $\varOmega_*^U$ *is a polynomial ring over* $\mathbb{Z}$ *with one generator* a_i *in every even dimension* $2i$, *where* $i \geq 1$.

Part 1 can be proved by the methods of Thom. Part 2 follows from the results of [3] and [4]. Part 3 is the most difficult one; it was done in 1960 in [4] (see also [5] for a more detailed account) and Milnor (unpublished, but see [9]).

Note that part 3 of Theorem 6.1 does not extend to complex bordism; $U_*(X)$ is not a free $\varOmega_*^U$-module in general. Unlike the case of unoriented bordism, the calculation of complex bordism of a space X does not reduce to calculating the coefficient ring $\varOmega_*^U$ and homology groups $H_*(X)$.

The calculation of the oriented bordism ring was completed by [4] (ring structure modulo torsion) and [10] (additive torsion), with important contributions made by Rokhlin, Averbuch, and Milnor. Unlike complex bordism, the ring $\varOmega_*^{SO}$ has additive torsion. We give only a partial result here, which does not fully describe the torsion elements. For the complete description of the ring $\varOmega_*^{SO}$ see the Oriented bordism page.

Theorem 6.3. (1) $\varOmega_*^{SO} \otimes \mathbb{Q}$ *is a polynomial ring over* $\mathbb{Q}$ *generated by the bordism classes of complex projective spaces* $\mathbb{C}P^{2i}$, $i \geq 1$.

(2) *The subring* $\mathrm{Tors} \subset \varOmega_*^{SO}$ *of torsion elements contains only elements of order 2. The quotient* $\varOmega_*^{SO}/\mathrm{Tors}$ *is a polynomial ring over* $\mathbb{Z}$ *with one generator* a_i *in every dimension* $4i$, *where* $i \geq 1$.

(3) *Two oriented manifolds are bordant if and only if they have identical sets of Pontrjagin and Stiefel-Whitney characteristic numbers.*

For more specific information about the three bordism theories, including constructions of manifolds representing polynomial generators in the bordism rings and applications, see the Unoriented bordism, Oriented bordism, and Complex bordism pages.

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