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Formal groups laws and genera*

TARAS PANOV

ABSTRACT. The article reviews role of formal group laws in bordism theory.

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1. Introduction

The theory of *formal group laws*, which originally appeared in algebraic geometry, was brought into bordism theory in the pioneering work [9]. The applications of formal group laws in algebraic topology are closely connected with *Hirzebruch genera* [6], which are important invariants of bordism classes of manifolds.

2. Elements of the theory of formal group laws

Let R be a commutative ring with unit.

A formal power series $F(u,v) \in R[[u,v]]$ is called a (commutative one-dimensional) *formal group law* over R if it satisfies the following equations:

(1) $F(u,0)=u,\ F(0,v)=v$;
(2) $F(F(u,v),w)=F(u,F(v,w))$;
(3) $F(u,v)=F(v,u)$.

The original example of a formal group law over a field $\mathbf{k}$ is provided by the expansion near the unit of the multiplication map $G\times G\to G$ in a one-dimensional algebraic group over $\mathbf{k}$. This also explains the terminology.

A formal group law F over R is called *linearisable* if there exists a coordinate change $u\mapsto g_F(u)=u+\sum_{i>1} g_iu^i \in R[[u]]$ such that

$$g_F(F(u,v))=g_F(u)+g_F(v).$$

Note that every formal group law over R determines a formal group law over $R\otimes\mathbb{Q}$.

Theorem 2.1. *Every formal group law F is linearisable over $R\otimes\mathbb{Q}$.*

Proof. Consider the series $\omega(u)=\frac{\partial F(u,w)}{\partial w}\Big|_{w=0}$. Then

$$\omega(F(u,v))=\frac{\partial F(F(u,v),w)}{\partial w}\Big|_{w=0}=\frac{\partial F(F(u,w),v)}{\partial F(u,w)}\cdot\frac{\partial F(u,w)}{\partial w}\Big|_{w=0}=\frac{\partial F(u,v)}{\partial u}\omega(u).$$

We therefore have $\frac{du}{\omega(u)}=\frac{dF(u,v)}{\omega(F(u,v))}$. Set

$$g(u)=\int_0^u\frac{dv}{\omega(v)};$$

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then $dg(u) = dg(F(u,v))$. This implies that $g(F(u,v)) = g(u)+C$. Since $F(0,v) = v$ and $g(0) = 0$, we get $C = g(v)$. Thus, $g(F(u,v)) = g(u) + g(v)$. □

A series $g(u) = u + \sum_{i>1} g_i u^i$ satisfying the equation $g(F(u,v)) = g(u) + g(v)$ is called a *logarithm* of the formal group law F; the above Theorem shows that a formal group law over $R \otimes \mathbb{Q}$ always has a logarithm. Its functional inverse series $f(t) \in R \otimes \mathbb{Q}[[t]]$ is called an *exponential* of the formal group law, so that we have $F(u,v) = f(g(u)+g(v))$ over $R \otimes \mathbb{Q}$. If R does not have torsion (i.e. $R \to R \otimes \mathbb{Q}$ is monic), the latter formula shows that a formal group law (as a series with coefficients in R) is fully determined by its logarithm (which is a series with coefficients in $R \otimes \mathbb{Q}$).

Let $F = \sum_{k,l} a_{kl} u^k v^l$ be a formal group law over a ring R and $r\colon R \to R'$ a ring homomorphism. Denote by $r(F)$ the formal series $\sum_{k,l} r(a_{kl}) u^k v^l \in R'[[u,v]]$; then $r(F)$ is a formal group law over R'.

A formal group law $\mathcal{F}$ over a ring A is *universal* if for any formal group law F over any ring R there exists a unique homomorphism $r\colon A \to R$ such that $F = r(\mathcal{F})$.

Proposition 2.2. *Assume that a universal formal group law $\mathcal{F}$ over A exists. Then*

(1) *The ring A is multiplicatively generated by the coefficients of the series $\mathcal{F}$;*
(2) *The universal formal group law is unique: if $\mathcal{F}'$ is another universal formal group law over A', then there is an isomorphism $r\colon A \to A'$ such that $\mathcal{F}' = r(\mathcal{F})$.*

Proof. To prove the first statement, denote by A' the subring in A generated by the coefficients of $\mathcal{F}$. Then there is a monomorphism $i\colon A' \to A$ satisfying $i(\mathcal{F}) = \mathcal{F}$. On the other hand, by universality there exists a homomorphism $r\colon A \to A'$ satisfying $r(\mathcal{F}) = \mathcal{F}$. It follows that $ir(\mathcal{F}) = \mathcal{F}$. This implies that $ir = \mathrm{id}\colon A \to A$ by the uniqueness requirement in the definition of $\mathcal{F}$. Thus $A' = A$. The second statement is proved similarly. □

Theorem 2.3 (Lazard [8])**.** *The universal formal group law $\mathcal{F}$ exists, and its coefficient ring A is isomorphic to the polynomial ring $\mathbb{Z}[a_1, a_2, \ldots]$ on an infinite number of generators.*

3. Formal group law of geometric cobordisms

The applications of formal group laws in cobordism theory build upon the following basic example.

Let X be a cell complex and $u, v \in U^2(X)$ two geometric cobordisms corresponding to elements $x, y \in H^2(X)$ respectively. Denote by $u +_H v$ the geometric cobordism corresponding to the cohomology class $x + y$.

Proposition 3.1. *The following relation holds in $U^2(X)$:*

$$u +_H v = F_U(u,v) = u + v + \sum_{k\geq 1,\, l\geq 1} \alpha_{kl}\, u^k v^l, \tag{1}$$

where the coefficients $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$ do not depend on X. The series $F_U(u,v)$ given by (1) *is a formal group law over the complex bordism ring Ω_U.*

Proof. We first do calculations with the universal example $X = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Then

$$U^*(\mathbb{C}P^\infty \times \mathbb{C}P^\infty) = \Omega_U^*[[\underline{u}, \underline{v}]],$$

where $\underline{u}, \underline{v}$ are canonical geometric cobordisms given by the projections of the space $\mathbb{C}P^\infty \times \mathbb{C}P^\infty$ onto its factors. We therefore have the following relation in the group $U^2(\mathbb{C}P^\infty \times \mathbb{C}P^\infty)$:

$$\underline{u} +_H \underline{v} = \sum_{k,l \geq 0} \alpha_{kl}\, \underline{u}^k \underline{v}^l, \tag{2}$$

where $\alpha_{kl} \in \Omega_U^{-2(k+l-1)}$.

Now let the geometric cobordisms $u, v \in U^2(X)$ be given by maps $f_u, f_v\colon X \to \mathbb{C}P^\infty$ respectively. Then $u = (f_u \times f_v)^*(\underline{u})$, $v = (f_u \times f_v)^*(\underline{v})$ and $u +_H v = (f_u \times f_v)^*(\underline{u} +_H \underline{v})$, where $f_u \times f_v\colon X \to \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. Applying the Ω_U^*-module map $(f_u \times f_v)^*$ to (2) we obtain the required formula (1). The fact that $F_U(u, v)$ is a formal group law follows directly from the properties of the group multiplication $\mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$. □

The series $F_U(u, v)$ is called the *formal group law of geometric cobordisms*; nowadays it is also usually referred to as the *"formal group law of complex cobordism"*.

The geometric cobordism $u \in U^2(X)$ is the first *Conner-Floyd Chern class* of the complex line bundle ξ over X obtained by pulling back the canonical bundle along the map $f_u\colon X \to \mathbb{C}P^\infty$. It follows that the formal group law of geometric cobordisms gives an expression of the first class $c_1^U(\xi \otimes \eta) \in U^2(X)$ of the tensor product of two complex line bundles over X in terms of the classes $u = c_1^U(\xi)$ and $v = c_1^U(\eta)$ of the factors:

$$c_1^U(\xi \otimes \eta) = F_U(u, v).$$

The coefficients of the formal group law of geometric cobordisms and its logarithm may be described geometrically by the following results.

Theorem 3.2 (Buchstaber [3]).

$$F_U(u, v) = \frac{\sum_{i,j \geq 0} [H_{ij}] u^i v^j}{\left(\sum_{r \geq 0} [\mathbb{C}P^r] u^r\right)\left(\sum_{s \geq 0} [\mathbb{C}P^s] v^s\right)},$$

where H_{ij} $(0 \leq i \leq j)$ are Milnor hypersurfaces and $H_{ji} = H_{ij}$.

Proof. Set $X = \mathbb{C}P^i \times \mathbb{C}P^j$ in Proposition 3.1. Consider the *Poincaré–Atiyah duality* map $D\colon U^2(\mathbb{C}P^i \times \mathbb{C}P^j) \to U_{2(i+j)-2}(\mathbb{C}P^i \times \mathbb{C}P^j)$ and the map $\varepsilon\colon U_*(\mathbb{C}P^i \times \mathbb{C}P^j) \to U_*(pt) = \Omega_*^U$ induced by the projection $\mathbb{C}P^i \times \mathbb{C}P^j \to pt$. Then the composition

$$\varepsilon D\colon U^2(\mathbb{C}P^i \times \mathbb{C}P^j) \to \Omega^U_{2(i+j)-2}$$

takes geometric cobordisms to the bordism classes of the corresponding submanifolds. In particular, $\varepsilon D(u +_H v) = [H_{ij}]$, $\varepsilon D(u^k v^l) = [\mathbb{C}P^{i-k}][\mathbb{C}P^{j-l}]$. Applying εD to (1) we obtain

$$[H_{ij}] = \sum_{k,\,l} \alpha_{kl} [\mathbb{C}P^{i-k}][\mathbb{C}P^{j-l}].$$

Therefore,

$$\sum_{i,j} [H_{ij}] u^i v^j = \Big(\sum_{k,\,l} \alpha_{kl} u^k v^l\Big)\Big(\sum_{i \geq k} [\mathbb{C}P^{i-k}] u^{i-k}\Big)\Big(\sum_{j \geq l} [\mathbb{C}P^{j-l}] v^{j-l}\Big),$$

which implies the required formula. □

Theorem 3.3 (Mishchenko, see [9]). *The logarithm of the formal group law of geometric cobordisms is given by the series*

$$g_U(u) = u + \sum_{k\ge1} \frac{[\mathbb{C}P^k]}{k+1} u^{k+1} \in \Omega_U \otimes \mathbb{Q}[[u]].$$

Proof. We have

$$dg_U(u) = \frac{du}{\left.\frac{\partial F_U(u,v)}{\partial v}\right|_{v=0}}.$$

Using the formula of Theorem 3.2 and the identity $H_{i0} = \mathbb{C}P^{i-1}$, we calculate

$$dg_U(u) = \frac{1 + \sum_{k>0}[\mathbb{C}P^k]u^k}{1 + \sum_{i>0}([H_{i1}] - [\mathbb{C}P^1][\mathbb{C}P^{i-1}])u^i}.$$

A calculation of Chern numbers shows that $[H_{i1}] = [\mathbb{C}P^1][\mathbb{C}P^{i-1}]$. Therefore, $dg_U(u) = 1 + \sum_{k>0}[\mathbb{C}P^k]u^k$, which implies the required formula. □

Using these calculations the following most important property of the formal group law F_U can be easily established:

Theorem 3.4 (Quillen [10]). *The formal group law F_U of geometric cobordisms is universal.*

Proof. Let $\mathcal{F}$ be the universal formal group law over a ring A. Then there is a homomorphism $r\colon A \to \Omega_U$ which takes $\mathcal{F}$ to F_U. The series $\mathcal{F}$, viewed as a formal group law over the ring $A \otimes \mathbb{Q}$, has the universality property for all formal group laws over $\mathbb{Q}$-algebras. Such a formal group law is determined by its logarithm, which is a series with leading term u. It follows that if we write the logarithm of $\mathcal{F}$ as $\sum b_k \frac{u^{k+1}}{k+1}$ then the ring $A \otimes \mathbb{Q}$ is the polynomial ring $\mathbb{Q}[b_1, b_2, \ldots]$. By Theorem 3.3, $r(b_k) = [\mathbb{C}P^k] \in \Omega_U$. Since $\Omega_U \otimes \mathbb{Q} \cong \mathbb{Q}[[\mathbb{C}P^1], [\mathbb{C}P^2], \ldots]$, this implies that $r \otimes \mathbb{Q}$ is an isomorphism.

By the Lazard Theorem the ring A does not have torsion, so r is a monomorphism. On the other hand, Theorem 3.2 implies that the image $r(A)$ contains the bordism classes $[H_{ij}] \in \Omega_U$, $0 \le i \le j$. Since these classes generate the whole ring Ω_U, the map r is onto and thus an isomorphism. □

The earliest applications of formal group laws in cobordism concerned finite group actions on manifolds, or "differentiable periodic maps", see [9], [2], [1]. For instance, a theorem of [9] describes the complex cobordism ring of the classifying space of the group $\mathbb{Z}/p$ as

$$U^*(B\mathbb{Z}/p) \cong \Omega_U[[u]]/[u]_p,$$

where $\Omega_U[[u]]$ denotes the ring of power series in one generator u of degree 2 with coefficients in Ω_U, and $[u]_p$ denotes the pth power in the formal group law of geometric cobordisms. This result extended and unified many earlier calculations of bordism with $\mathbb{Z}/p$-actions from [4].

The universality of the formal group law of geometric cobordisms has important consequences for the stable homotopy theory: it implies that complex bordism is the universal complex oriented homology theory.

4. Hirzebruch genera

Every homomorphism $\varphi\colon \Omega_U \to R$ from the complex cobordism ring to a commutative ring R with unit can be regarded as a multiplicative characteristic of manifolds which is an invariant of cobordism classes. Such a homomorphism is called a (complex) *R-genus*. (The term *'multiplicative genus'* is also used, to emphasise that such a genus is a ring homomorphism; in classical algebraic geometry, there are instances of genera which are not multiplicative.)

Assume that the ring R does not have additive torsion. Then every R-genus φ is fully determined by the corresponding homomorphism $\Omega_U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$, which we shall also denote by φ. The following famous construction of [6] allows us to describe homomorphisms $\varphi\colon \Omega_U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$ by means of universal R-valued characteristic classes of special type.

4.1. **Construction.** Let $BU = \lim_{n\to\infty} BU(n)$. Then $H^*(BU)$ is isomorphic to the graded ring of formal power series $\mathbb{Z}[[c_1, c_2, \ldots]]$ in universal Chern classes, $\deg c_k = 2k$. The set of Chern characteristic numbers of a manifold M defines an element in $\mathrm{Hom}(H^*(BU), \mathbb{Z})$, which in fact belongs to the subgroup $H_*(BU)$ in the latter group. We therefore obtain a group homomorphism

$$\Omega_U \to H_*(BU).$$

Since the multiplication in the ring $H_*(BU)$ is obtained from the maps $BU_k \times BU_l \to BU_{k+l}$ corresponding to the Whitney sum of vector bundles, and the Chern classes have the appropriate multiplicative property, $\Omega_U \to H_*(BU)$ is a ring homomorphism.

Part 2 of the structure theorem for complex bordism says that $\Omega_U \to H_*(BU)$ is a monomorphism, and Part 1 of the same theorem says that the corresponding $\mathbb{Q}$-map $\Omega_U \otimes \mathbb{Q} \to H_*(BU; \mathbb{Q})$ is an isomorphism. It follows that every homomorphism $\varphi\colon \Omega_U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$ can be interpreted as an element of

$$\mathrm{Hom}_{\mathbb{Q}}(H_*(BU; \mathbb{Q}), R \otimes \mathbb{Q}) = H^*(BU; \mathbb{Q}) \otimes R,$$

or as a sequence of homogeneous polynomials $\{K_i(c_1, \ldots, c_i),\ i \geq 0\}$, $\deg K_i = 2i$. This sequence of polynomials cannot be chosen arbitrarily; the fact that φ is a ring homomorphism imposes certain conditions. These conditions may be described as follows: an identity

$$1 + c_1 + c_2 + \cdots = (1 + c_1' + c_2' + \cdots) \cdot (1 + c_1'' + c_2'' + \cdots)$$

implies the identity

$$\sum_{n\geq 0} K_n(c_1, \ldots, c_n) = \sum_{i\geq 0} K_i(c_1', \ldots, c_i') \cdot \sum_{j\geq 0} K_j(c_1'', \ldots, c_j'').$$

A sequence of homogeneous polynomials $K = \{K_i(c_1, \ldots, c_i), i \geq 0\}$ with $K_0 = 1$ satisfying these identities is called a *multiplicative Hirzebruch sequence.*

Such a multiplicative sequence K is completely determined by the series $Q(x) = 1 + q_1 x + q_2 x^2 + \cdots \in R \otimes \mathbb{Q}[[x]]$, where $x = c_1$, and $q_i = K_i(1, 0, \ldots, 0)$; moreover, every series $Q(x)$ as above determines a multiplicative sequence. Indeed, by considering the identity

$$1 + c_1 + \cdots + c_n = (1 + x_1) \cdots (1 + x_n)$$

we obtain that

$$Q(x_1) \cdots Q(x_n) = 1+K_1(c_1)+K_2(c_1, c_2)+\cdots+K_n(c_1, \dots, c_n)+K_{n+1}(c_1, \dots, c_n, 0)+\cdots .$$

Along with the series $Q(x)$ it is convenient to consider the series $f(x) \in R \otimes \mathbb{Q}[[x]]$ given by the identity

$$Q(x) = \frac{x}{f(x)}; \quad f(x) = x + f_1 x + f_2 x^2 + \cdots .$$

It follows that the ring homomorphisms $\varphi\colon \varOmega_U \otimes \mathbb{Q} \to R \otimes \mathbb{Q}$ are in one-to-one correspondence with the series $f(x) \in R \otimes \mathbb{Q}[[x]]$. Under this correspondence, the value of φ on an $2n$-dimensional bordism class $[M] \in \varOmega_U$ is given by

$$\varphi[M] = \Bigl(\prod_{i=1}^{n} \frac{x_i}{f(x_i)}, \langle M\rangle\Bigr)$$

where one needs to plug in the Chern classes $c_1, \dots, c_n$ for the elementary symmetric functions in $x_1, \dots, x_n$ and then calculate the value of the resulting characteristic class on the fundamental class $\langle M\rangle \in H_{2n}(M)$.

The homomorphism $\varphi\colon \varOmega_U \to R \otimes \mathbb{Q}$ given by the formula above is called the *Hirzebruch genus* associated to the series $f(x) = x + f_1 x + f_2 x^2 + \cdots \in R \otimes \mathbb{Q}[[x]]$. Thus, there is a one-two-one correspondence between series $f(x) \in R\otimes\mathbb{Q}[[x]]$ having leading term x and genera $\varphi\colon \varOmega_U \to R \otimes \mathbb{Q}$.

We shall also denote the characteristic class $\prod_{i=1}^{n} \frac{x_i}{f(x_i)}$ of a complex vector bundle ξ by $\varphi(\xi)$; so that $\varphi[M] = \varphi(\mathcal{T}M)\langle M\rangle$.

4.2. **Connection to formal group laws.** Every genus $\varphi\colon \varOmega_U \to R$ gives rise to a formal group law $\varphi(F_U)$ over R, where F_U is the formal group law of geometric cobordisms.

Theorem 4.1. *For every genus $\varphi\colon \varOmega_U \to R \otimes \mathbb{Q}$, the exponential of the formal group law $\varphi(F_U)$ is given by the series $f(x) \in R \otimes \mathbb{Q}[[x]]$ corresponding to φ.*

1st proof. Let X be a manifold and $u, v \in U^2(X)$ its two geometric cobordisms defined by the elements $x, y \in H^2(X)$ respectively. By the definition of the formal group law $F_U(u, v) = u+v+\sum_{k\ge1,l\ge1} \alpha_{kl} u^k v^l$ we have the following relation between geometric cobordisms in $U^2(X)$:

$$[M_{x+y}] = \sum_{k,l\ge0} \alpha_{kl}[M_{x^k y^l}]$$

in $\varOmega_U$, where $M_{x+y} \subset X$ is the codimension 2 submanifold dual to $x + y \in H^2(X)$, and $M_{x^k y^l} \subset X$ is a codimension $2(k + l)$ submanifold dual to $x^k y^l \in H^2(X)$. Applying the genus φ we obtain

$$\varphi[M_{x+y}] = \sum \varphi(\alpha_{kl})\varphi[M_{x^k y^l}]. \tag{3}$$

Let $\iota\colon M_{x+y} \subset X$ be the embedding. Considering the decomposition

$$\iota^*(\mathcal{T}X) = \mathcal{T}M_{x+y} \oplus \nu(\iota)$$

and using the multiplicativity of the characteristic class φ we obtain

$$\iota^*\varphi(\mathcal{T}X) = \varphi(\mathcal{T}M_{x+y}) \cdot \iota^*(\tfrac{x+y}{f(x+y)}).$$

Therefore,

$$\varphi[M_{x+y}] = \iota^*\Big(\varphi(\mathcal{T}X) \cdot \tfrac{f(x+y)}{x+y}\Big)\langle M_{x+y}\rangle = \Big(\varphi(\mathcal{T}X) \cdot f(x+y)\Big)\langle X\rangle. \tag{4}$$

Similarly, by considering the embedding $M_{x^k y^l} \to X$ we obtain

$$\varphi[M_{x^k y^l}] = \Big(\varphi(\mathcal{T}X) \cdot f(x)^k f(y)^l\Big)\langle X\rangle. \tag{5}$$

Plugging (4) and (5) into (3) we finally obtain

$$f(x+y) = \sum_{k,l\ge 0} \varphi(\alpha_{kl}) f(x)^k f(y)^l.$$

This implies, by definition, that f is the exponential of $\varphi(F_U)$. □

2nd proof. The complex bundle isomorphism $\mathcal{T}(\mathbb{C}P^k) \oplus \underline{\mathbb{C}} = \bar\eta \oplus \ldots \oplus \bar\eta$ ($k+1$ summands) allows us to calculate the value of a genus on $\mathbb{C}P^k$ explicitly. Let $x = c_1(\bar\eta) \in H^2(\mathbb{C}P^k)$ and let g be the series functionally inverse to f; then

$$\begin{aligned}\varphi[\mathbb{C}P^k] &= \Big(\frac{x}{f(x)}\Big)^{k+1}\langle \mathbb{C}P^k\rangle\\ &= \text{coefficient of } x^k \text{ in } \Big(\frac{x}{f(x)}\Big)^{k+1} = \operatorname{res}_0\Big(\frac{1}{f(x)}\Big)^{k+1}\\ &= \frac{1}{2\pi i}\oint\Big(\frac{1}{f(x)}\Big)^{k+1}dx = \frac{1}{2\pi i}\oint\frac{1}{u^{k+1}}g'(u)du\\ &= \operatorname{res}_0\Big(\frac{g'(u)}{u^{k+1}}\Big) = \text{coefficient of } u^k \text{ in } g'(u).\end{aligned}$$

(Integrating over a closed path around zero makes sense only for convergent power series with coefficients in $\mathbb{C}$, however the result holds for all power series with coefficients in $R \otimes \mathbb{Q}$.) Therefore,

$$g'(u) = \sum_{k\ge 0} \varphi[\mathbb{C}P^k]u^k.$$

This implies that g is the logarithm of the formal group law $\varphi(F_U)$, and thus f is its exponential. □

A parallel theory of genera exists for oriented manifolds. These genera are homomorphisms $\Omega_{SO} \to R$ from the oriented bordism ring, and the Hirzebruch construction expresses genera over $\mathbb{Q}$-algebras via certain Pontrjagin characteristic classes (which replace the Chern classes).

4.3. **Examples.** We take $R = \mathbb{Z}$ in these examples:

(1) The top Chern number $c_n(\xi)[M]$ is a Hirzebruch genus, and its corresponding f-series is $f(x) = \frac{x}{1+x}$. The value of this genus on a stably complex manifold $(M, c_\mathcal{T})$ equals the Euler characteristic of M if $c_\mathcal{T}$ is an *almost* complex structure.

(2) The *L-genus* $L[M]$ corresponds to the series $f(x) = \tanh(x)$ (the hyperbolic tangent). It is equal to the signature of M by the classical Hirzebruch formula [6].
(3) The *Todd genus* $\mathrm{td}[M]$ corresponds to the series $f(x) = 1 - e^{-x}$. It takes value 1 on every complex projective space $\mathbb{C}P^k$.

The 'trivial' genus $\varepsilon\colon \Omega_U \to \mathbb{Z}$ corresponding to the series $f(x) = x$ gives rise to the *augmentation transformation* $U^* \to H^*$ from complex cobordism to ordinary cohomology (also known as the *Thom homomorphism*). More generally, for every genus $\varphi\colon \Omega_U \to R$ and a space X we may set $h^*_\varphi(X) = U^*(X) \otimes_{\Omega_U} R$. Under certain conditions guaranteeing the exactness of the sequences of pairs (known as the *Landweber exact functor theorem* [7]) the functor $h^*_\varphi(\cdot)$ gives rise to a complex-oriented cohomology theory with the coefficient ring R.

As an example of this procedure, consider a formal indeterminate β of degree -2, and let $f(x) = 1 - e^{-\beta x}$. The corresponding genus, which is also called the *Todd genus*, takes values in the ring $\mathbb{Z}[\beta]$. By interpreting β as the *Bott element* in the complex K-group $\widetilde{K}^0(S^2) = K^{-2}(pt)$ we obtain a homomorphism $\mathrm{td}\colon \Omega^*_U \to K^*(pt)$. It gives rise to a multiplicative transformation $U^* \to K^*$ from complex cobordism to complex K-theory introduced by Conner and Floyd [5]. In this paper Conner and Floyd proved that complex cobordism determines complex K-theory by means of the isomorphism $K^*(X) \cong U^*(X) \otimes_{\Omega_U} \mathbb{Z}[\beta]$, where the Ω_U-module structure on $\mathbb{Z}[\beta]$ is given by the Todd genus. Their proof makes use of the Conner-Floyd Chern classes; several proofs were given subsequently, including one which follows directly from the Landweber exact functor theorem.

Another important example from the original work of Hirzebruch is given by the χ_y*-genus*. It corresponds to the series

$$f(x) = \frac{1 - e^{-x(1+y)}}{1 + ye^{-x(1+y)}},$$

where $y \in \mathbb{R}$ is a parameter. Setting $y = -1$, $y = 0$ and $y = 1$ we get the top Chern number $c_n[M]$, the Todd genus $\mathrm{td}[M]$ and the L-genus $L[M] = \mathrm{sign}(M)$ respectively.

If M is a complex manifold then the value $\chi_y[M]$ can be calculated in terms of the Euler characteristics of Dolbeault complexes on M.

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TARAS PANOV
DEPARTMENT OF GEOMETRY AND TOPOLOGY
FACULTY OF MATHEMATICS AND MECHANICS
MOSCOW STATE UNIVERSITY, LENINSKIE GORY
119991 MOSCOW, RUSSIA

E-mail address: panov@higeom.math.msu.su
Web address: http://higeom.math.msu.su/people/taras/english.html