

## Simplicial volume\*

CLARA LÖH

ABSTRACT. Simplicial volume is a homotopy invariant of oriented closed connected manifolds introduced by Gromov in his proof of Mostow rigidity. On the one hand, the simplicial volume of a Riemannian manifold encodes non-trivial information about the Riemannian volume; on the other hand, simplicial volume can be described in terms of a certain functional analytic version of homological algebra (bounded cohomology). In this article we survey important properties and applications of simplicial volume as well as useful techniques for working with simplicial volume.

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### 1. DEFINITION AND HISTORY

*Simplicial volume* is a homotopy invariant of oriented closed connected **manifolds** that was introduced by Gromov in his proof of Mostow rigidity [31][13]. Intuitively, simplicial volume measures how difficult it is to describe the manifold in question in terms of simplices (with real coefficients):

**Definition 1.1.** (Simplicial volume) Let  $M$  be an oriented closed connected manifold of dimension  $n$ . Then the **simplicial volume** of  $M$  (also called the **Gromov norm** of  $M$ ) is defined as

$$\|M\| := \|[M]\|_1 = \inf \left\{ |c|_1 \mid c \in C_n(M; \mathbb{R}) \text{ is a fundamental cycle of } M \right\} \in \mathbb{R}_{\geq 0},$$

where  $[M] \in H_n(M; \mathbb{R})$  is the fundamental class of  $M$  with real coefficients.

- Here,  $|\cdot|_1$  denotes the  $\ell^1$ -norm on the singular chain complex  $C_*(\cdot; \mathbb{R})$  with real coefficients induced from the (unordered) basis given by all singular simplices, i.e.: for a topological space  $X$  and a chain  $c = \sum_{j=0}^k a_j \cdot \sigma_j \in C_*(X; \mathbb{R})$  (in reduced form), the  $\ell^1$ -**norm** of  $c$  is given by

$$|c|_1 := \sum_{j=0}^k |a_j|.$$

- Moreover,  $\|\cdot\|_1$  denotes the  $\ell^1$ -**semi-norm** on singular homology  $H_*(\cdot; \mathbb{R})$  with real coefficients, which is induced by  $|\cdot|_1$ . More explicitly, if  $X$  is a topological space and  $\alpha \in H_*(X; \mathbb{R})$ , then

$$\|\alpha\|_1 := \inf \left\{ |c|_1 \mid c \in C_*(X; \mathbb{R}) \text{ is a cycle representing } \alpha \right\}.$$

**Convention 1.2.** In the following, if not explicitly stated otherwise, all manifolds are topological manifolds and are of non-zero dimension.

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\*Atlas page: [www.map.mpim-bonn.mpg.de/Simplicial\\_volume](http://www.map.mpim-bonn.mpg.de/Simplicial_volume)

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## 2. FUNCTORIALITY AND ELEMENTARY EXAMPLES

The  $\ell^1$ -semi-norm is functorial in the following sense [14, Section 5.34]:

**Proposition 2.1** (Functoriality of the  $\ell^1$ -semi-norm). *If  $f: X \rightarrow Y$  is a continuous map of topological spaces and  $\alpha \in H_*(X; \mathbb{R})$ , then*

$$\|H_*(f; \mathbb{R})(\alpha)\|_1 \leq \|\alpha\|_1,$$

as can be seen by inspecting the definition of  $H_*(f; \mathbb{R}) = H_*(C_*(f; \mathbb{R}))$  and of  $\|\cdot\|_1$ .

**Corollary 2.2.** • *Let  $f: M \rightarrow N$  be a map of oriented closed connected manifolds of the same dimension. Then*

$$|\deg f| \cdot \|N\| \leq \|M\|.$$

- *Because homotopy equivalences of oriented closed connected manifolds have degree  $-1$  or  $1$ , it follows that simplicial volume indeed is a homotopy invariant of oriented closed connected manifolds.*

Hence, all oriented closed connected manifolds admitting a self-map of non-trivial degree (i.e., not equal to  $-1$ ,  $0$ , or  $1$ ) have vanishing simplicial volume; for instance, the simplicial volume of all

- spheres
- tori
- (odd-dimensional) real projective spaces
- complex projective spaces

is zero. Similarly, any oriented closed connected smooth manifold that admits a non-trivial smooth  $S^1$ -action has vanishing simplicial volume [36].

## 3. ‘COMPUTING’ SIMPLICIAL VOLUME

In most cases, trying to compute simplicial volume by inspecting the definition proves to be futile; the two main sources for non-trivial estimates and inheritance properties of simplicial volume are:

- *Geometric approach:* The connection between simplicial volume and Riemannian geometry (see Section 3.1 below).
- *Algebraic approach:* The connection between simplicial volume and bounded cohomology (see Section 3.2 below).

**3.1. Simplicial volume and Riemannian geometry.** A fascinating aspect of simplicial volume is that it is a *homotopy* invariant encoding non-trivial information about the Riemannian volume. The most fundamental result of this type is Gromov’s Main Inequality [13, Section 0.5] and the resulting lower bound of the minimal volume in terms of the simplicial volume [13, Section 0.5] (the improved upper bound is due to Besson, Courtois, and Gallot [4, Théorème D]):

**Theorem 3.1** (Main Inequality). *For all oriented closed connected Riemannian  $n$ -manifolds  $M$  whose Ricci curvature is bounded from below by  $-1/(n-1)$  we have*

$$\|M\| < n! \cdot \text{vol}(M).$$

**Theorem 3.2** (Simplicial volume and minimal volume). *For all oriented closed connected smooth  $n$ -manifolds  $M$  we have*

$$\|M\| \leq \frac{(n-1)^n \cdot n!}{n^{n/2}} \cdot \text{minvol}(M).$$

The **minimal volume** [13] of a complete smooth manifold  $M$  is defined as  $\text{minvol}(M) := \inf\{\text{vol}(M, g) \mid g \text{ is a Riemannian metric on } M \text{ with } |\text{sec}(g)| \leq 1\}$ .

Conversely, in the presence of negative curvature, the simplicial volume is bounded from below by the Riemannian volume [13, Section 0.3][35, Theorem 6.2][16]:

**Theorem 3.3** (Simplicial volume and negative sectional curvature).

- *The simplicial volume of oriented closed connected Riemannian manifolds of negative sectional curvature is non-zero. More precisely: For every  $n \in \mathbb{N}$  there is a constant  $C_n \in \mathbb{R}_{>0}$  such that the following holds: If  $M$  is an oriented closed connected Riemannian  $n$ -manifold whose sectional curvature is bounded from above by  $\delta \in \mathbb{R}_{<0}$ , then*

$$\|M\| > C_n \cdot |\delta|^{n/2} \cdot \text{vol}(M).$$

- *Let  $M$  be an oriented closed connected hyperbolic  $n$ -manifold. Then  $\|M\| = \text{vol}(M)/v_n$ , where  $v_n$  is the supremal volume of all geodesic  $n$ -simplices in hyperbolic  $n$ -space (indeed,  $v_n$  is finite [35, Proposition 6.1.4]).*

The *proof* of the lower bound  $\|\cdot\| \geq \text{const} \cdot \text{vol}(\cdot)$  consists of ‘straightening’ fundamental cycles to cycles consisting only of singular simplices whose lifts to the Riemannian universal covering are geodesic.

It is well known that  $v_2 = \pi$  [35, p.6.3], and hence, for any oriented closed connected surface  $S_g$  of genus  $g \in \mathbb{N}_{\geq 1}$  we have  $\|S_g\| = 4 \cdot g - 4$ .

Generalisations of Theorem 3.3 are:

- The proportionality principle for simplicial volume (Theorem 4.4);
- the non-vanishing of the simplicial volume of oriented closed connected locally symmetric spaces of non-compact type [19] (which is obtained by a combination of a generalised straightening and the proportionality principle);
- non-vanishing results for certain manifolds with negatively curved fundamental group (see Section 5.3 below);
- the construction of (aspherical) oriented closed connected manifolds of non-zero simplicial volume via (relative) hyperbolisation techniques [1].

**3.2. Simplicial volume and bounded cohomology.** A more algebraic approach to simplicial volume is based on the following observation [13, p.17][2, F.2.2] (see below for an explanation of the notation):

**Proposition 3.4** (Duality principle). *Let  $X$  be a topological space, let  $n \in \mathbb{N}$ , and let  $\alpha \in H_n(X; \mathbb{R})$ . Then*

$$\begin{aligned} \|\alpha\|_1 &= \sup\left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H^n(X; \mathbb{R}), \langle \varphi, \alpha \rangle = 1 \right\} \\ &= \sup\left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H_b^n(X; \mathbb{R}), \langle \varphi, \alpha \rangle = 1 \right\}. \end{aligned}$$

**Corollary 3.5.** *Let  $M$  be an oriented closed connected  $n$ -manifold. Then*

$$\begin{aligned} \|M\| &= \frac{1}{\|[M]^*\|_\infty} \\ &= \sup \left\{ \frac{1}{\|\varphi\|_\infty} \mid \varphi \in H_b^n(M; \mathbb{R}), c_M(\varphi) = [M]^* \right\} \end{aligned}$$

where  $[M]^* \in H^n(M; \mathbb{R})$  denotes the cohomology class dual to the real fundamental class of  $M$ .

For the sake of completeness, we review the definition of *bounded cohomology* of topological spaces:

**Definition 3.6.** (Bounded cohomology) Let  $X$  be a topological space, and let  $n \in \mathbb{N}$ .

- If  $f \in C^n(X; \mathbb{R})$  is a cochain, then we write

$$|f|_\infty := \sup_{\sigma \in \text{map}(\Delta^n, X)} |f(\sigma)| \in \mathbb{R}_{\geq 0} \cup \{\infty\}.$$

If  $|f|_\infty < \infty$ , then  $f$  is a **bounded cochain**.

- We write  $C_b^n(X; \mathbb{R}) := \{f \mid f \in C^n(X; \mathbb{R}), |f|_\infty < \infty\}$  for the subspace of bounded cochains. Notice that  $C_b^n(X; \mathbb{R})$  is a subcomplex of the singular cochain complex, called the **bounded cochain complex** of  $X$ .
- The cohomology  $H_b^*(X; \mathbb{R})$  of  $C_b^*(X; \mathbb{R})$  is the **bounded cohomology** of  $X$ . (It is not difficult to see that every continuous map induces a homomorphism on the level of bounded cohomology; turning  $H_b^*(\cdot; \mathbb{R})$  into a contravariant functor).
- The norm  $|\cdot|_\infty$  on the bounded cochain complex induces a semi-norm on bounded cohomology: If  $\varphi \in H_b^n(X; \mathbb{R})$ , then

$$\|\varphi\|_\infty := \inf \{ |f|_\infty \mid f \in C_b^n(X; \mathbb{R}) \text{ is a cocycle representing } \varphi \}.$$

- The natural inclusion  $C_b^*(X; \mathbb{R}) \hookrightarrow C^*(X; \mathbb{R})$  induces a natural homomorphism  $c_X: H_b^*(X; \mathbb{R}) \longrightarrow H^*(X; \mathbb{R})$ , the **comparison map**.

Bounded cohomology was originally introduced by Trauber. Gromov further developed bounded cohomology and studied its relation with the (Riemannian) volume of manifolds [13]. A more algebraic approach to bounded cohomology was subsequently developed by Brooks [7], Ivanov [17], Noskov [33], Monod [28][29], and Bühler [11].

In the context of simplicial volume, bounded cohomology contributed to establish vanishing results (Section 5.2) in the presence of amenable fundamental groups, non-vanishing results (Section 5.3) in the presence of certain types of negative curvature, and inheritance properties with respect to products (Section 4.2), connected sums (Section 4.3) and shared Riemannian coverings (Section 4.5).

## 4. INHERITANCE PROPERTIES

**4.1. Finite Coverings.** Simplicial volume is multiplicative with respect to finite coverings:

**Proposition 4.1** (Simplicial volume and finite coverings). *Let  $p: M \rightarrow N$  be a covering map of oriented closed connected manifolds, and let  $d$  be the (finite!) number of sheets of  $p$ . Then*

$$\|M\| = d \cdot \|N\|.$$

For the *proof* notice that  $d = |\deg p|$  and that  $\|M\| \geq |\deg p| \cdot \|N\|$  by functoriality (Corollary 2.2); to show the converse inequality  $\|M\| \leq d \cdot \|N\|$  one can use the transfer on singular chains.

**4.2. Products.** Simplicial volume is almost multiplicative with respect to direct products of manifolds [13, p. 10][2, Theorem F.2.5]:

**Theorem 4.2** (Simplicial volume and products). *Let  $M$  and  $N$  be oriented closed connected manifolds. Then*

$$\|M\| \cdot \|N\| \leq \|M \times N\| \leq \binom{\dim(M) + \dim(N)}{\dim(M)} \cdot \|M\| \cdot \|N\|.$$

A *proof* of the right hand estimate can be given by looking at the concrete description of  $[M \times N] = [M] \times [N]$  in terms of the cross-product of singular chains; a proof of the left hand estimate can be obtained by using the duality principle (Corollary 3.5) and the fact that the norm  $\|\cdot\|_\infty$  is submultiplicative with respect to the cross-product of (bounded) singular cochains.

Notice that simplicial volume in general is not multiplicative: Bucher-Karlsson [8, Corollary 2] proved that  $\|S \times S'\| = 3/2 \cdot \|S\| \cdot \|S'\|$  holds for all oriented closed connected surfaces  $S, S'$  of genus at least 2 (and  $\|S\| \neq 0 \neq \|S'\|$  (see Section 3.1 above)).

**4.3. Connected sums.** Simplicial volume is additive with respect to connected sums in the following sense [13, p. 10]:

**Theorem 4.3** (Simplicial volume and connected sums). *Let  $M$  and  $N$  be oriented closed connected manifolds of dimension at least 3. Then*

$$\|M \# N\| = \|M\| + \|N\|.$$

Notice that simplicial volume in general is *not* additive with respect to connected sums in dimension 2: The simplicial volume of the torus is zero (see Section 2 above), but the simplicial volume of an oriented closed connected surface of genus 2 is non-zero (see Section 3.1 above) is non-zero.

The *proof* of Theorem 4.3 is based on the mapping theorem in bounded cohomology (Theorem 5.1) and a careful analysis of so-called tree-like complexes [13, Section 3.5]. Generalising these arguments, it can be seen that also additivity for simplicial volume with respect to certain ‘amenable’ gluings holds [18].

**4.4. Fibre bundles.**

- In low dimensions, there is a relation between the simplicial volume of the total space of a fibre bundle of oriented closed connected manifolds and the product of the simplicial volume of base and fibre [15][10].

- However, in general, the simplicial volume of a fibre bundle of oriented closed connected manifolds is *not* related in an obvious way to the simplicial volume of base and fibre [15]: There exist oriented closed connected hyperbolic 3-manifolds that fibre over the circle. However, the circle has simplicial volume equal to zero, while the simplicial volume of the hyperbolic 3-manifold in question is non-zero (Theorem 3.3).
- In certain cases, amenable fibres force the simplicial volume to vanish (see Section 5.2 below).

**4.5. Proportionality principle.** For hyperbolic manifolds the simplicial volume is proportional to the Riemannian volume. Gromov and Thurston generalised this result suitably to cover all Riemannian manifolds:

**Theorem 4.4** (Proportionality principle). *Let  $M$  and  $N$  be oriented closed connected Riemannian manifolds that have isometric Riemannian universal coverings. Then*

$$\frac{\|M\|}{\text{vol}(M)} = \frac{\|N\|}{\text{vol}(N)}.$$

Both Gromov's and Thurston's *proof* of this result make use of an averaging process. More precisely:

- *Gromov's strategy:* Use the duality principle (Corollary 3.5) and average (bounded) continuous singular cochains over the isometry group of the Riemannian universal covering modulo the fundamental group; this requires a careful analysis of the relation between (bounded) continuous singular cohomology and (bounded) singular cohomology [13, Section 2.3][9][12].
- *Thurston's strategy:* Replace singular homology by measure homology, and average/smear measure chains over the isometry group of the Riemannian universal covering; this requires a careful analysis of the relation between measure homology and singular homology [35, p. 6.9][22][23].

## 5. SIMPLICIAL VOLUME AND THE FUNDAMENTAL GROUP

In view of the duality principle (Corollary 3.5) we can interpret simplicial volume in terms of bounded cohomology. The key to deducing interesting consequences for simplicial volume from bounded cohomology is the relation of bounded cohomology to the fundamental group via the mapping theorem (Theorem 5.1), and the relation of bounded cohomology to geometric group theory (see Section 5.3 below).

**5.1. Background: Mapping theorem in bounded cohomology.** One of the most fundamental (and most surprising) features of bounded cohomology is that it cannot detect **amenable groups** in the homotopy groups of a space [13, Section 3.1][17]:

**Theorem 5.1** (Mapping theorem in bounded cohomology). *Let  $f: X \rightarrow Y$  be a (base-point preserving) continuous map between connected countable CW-complexes such that  $\pi_1(f): \pi_1(X) \rightarrow \pi_1(Y)$  is surjective and has amenable kernel.*

- *Then the induced map  $H_b^*(f; \mathbb{R}): H_b^*(Y; \mathbb{R}) \rightarrow H_b^*(X; \mathbb{R})$  in bounded cohomology is an isometric isomorphism.*

- In particular, the map  $H_*(f; \mathbb{R}): H_*(X; \mathbb{R}) \longrightarrow H_*(Y; \mathbb{R})$  is isometric with respect to the  $\ell^1$ -semi-norm  $\|\cdot\|_1$ . (In general, this does not say anything about injectivity/surjectivity of  $H_*(f; \mathbb{R})$ ).

Notice however that simplicial volume does not depend only on the fundamental group, but rather on the classifying map to  $B\pi_1(\cdot)$ ; for instance, any oriented closed connected hyperbolic manifold  $M$  satisfies

$$\|M\| \neq 0 = 0 \cdot \|M\| = \|S^2 \times M\| \quad \text{and} \quad \pi_1(M) \cong \pi_1(S^2 \times M).$$

More generally, if  $M$  is an oriented closed connected  $n$ -manifold whose fundamental group has (rational) cohomological dimension less than  $n$ , then  $\|M\| = 0$ .

**5.2. Amenability and vanishing results.** Direct consequences of the mapping theorem for bounded cohomology (Theorem 5.1) are:

- The simplicial volume of oriented closed connected manifolds with amenable fundamental group is zero;
- this includes, in particular, the case of trivial, Abelian, solvable, and nilpotent fundamental groups.
- Let  $M$  be an oriented closed connected manifold that is the total space of a fibration whose base and fibre are oriented closed connected manifolds (of non-zero dimension), and where the fibre has amenable fundamental group. Then  $\|M\| = 0$  [25, Exercise 14.15 and p. 556].

More generally: A stronger vanishing result for the comparison map  $H_b^*(\cdot; \mathbb{R}) \longrightarrow H^*(\cdot; \mathbb{R})$  for spaces with amenable coverings of small multiplicity, [13, Vanishing Finiteness Theorem on p. 58][17, Corollary 6.3], leads to the following statement: If  $M$  is an oriented closed connected  $n$ -manifold that admits a covering by amenable open subsets of multiplicity at most  $n$ , then  $\|M\| = 0$ .

**5.3. Hyperbolicity and non-vanishing results.** The non-vanishing of the simplicial volume of Riemannian manifolds with negative sectional curvature (Theorem 3.3) together with the duality principle implies that the cohomological fundamental class of such a manifold lies in the image of the comparison map  $H_b^*(\cdot; \mathbb{R}) \longrightarrow H^*(\cdot; \mathbb{R})$ . More generally, Mineyev showed that (in the case of rationally essential manifolds) the condition of negative sectional curvature can be relaxed to [word hyperbolicity](#) of the fundamental group [26][27]:

**Theorem 5.2** (Bounded cohomology and word hyperbolic groups). *Let  $G$  be a finitely presented group. Then the following are equivalent:*

- The group  $G$  is word hyperbolic.
- The comparison map  $H_b^*(BG; V) \longrightarrow H^*(BG; V)$  is surjective for all  $k \in \mathbb{N}_{\geq 2}$  and all Banach  $G$ -modules  $V$ .

**Corollary 5.3** (Simplicial volume and word hyperbolic fundamental groups). *Let  $M$  be an oriented closed connected manifold of dimension at least 2 that is rationally essential (e.g., aspherical). If the fundamental group of  $M$  is word hyperbolic, then  $\|M\| > 0$ .*

## 6. VARIATIONS OF SIMPLICIAL VOLUME

For simplicity, in the previous sections we only considered the case that all manifolds in question were orientable and closed (and connected). In the following, we briefly list possible extensions of the definition of simplicial volume to larger classes of manifolds.

**6.1. Simplicial volume of manifolds with boundary.** For oriented connected compact  $n$ -manifolds  $(M, \partial M)$  with boundary, the **relative simplicial volume**  $\|M, \partial M\|$  is defined as the  $\ell^1$ -semi-norm on relative singular homology  $H_n(M, \partial M; \mathbb{R})$  of the relative fundamental class  $[M, \partial M] \in H_n(M, \partial M; \mathbb{R})$  [13][18]; the extension of simplicial volume to manifolds with boundary plays an important role in the theory of 3-manifolds (see Section 7 below).

**6.2. Simplicial volume of non-compact manifolds.** Non-compact, oriented, connected manifolds have a fundamental class in locally finite homology; notice that there is a (possibly infinite)  $\ell^1$ -semi-norm on locally finite homology with real coefficients. There are two main flavours of simplicial volumes of non-compact manifolds – a topological one and a geometric one (where a Lipschitz condition is imposed on the fundamental cycles):

(1) *Topological version* [13]: Let  $M$  be an oriented connected  $n$ -manifold without boundary, and let  $[M] \in H_n^{\text{lf}}(M; \mathbb{R})$  be its locally finite fundamental class (with real coefficients). Then the **simplicial volume**  $\|M\|$  of  $M$  is defined as the  $\ell^1$ -semi-norm of  $[M]$ .

- For all proper continuous maps  $f: M \rightarrow N$  between oriented connected manifolds without boundary of the same dimension we have the inequality  $|\deg f| \cdot \|N\| \leq \|M\|$ .
- Notice that the simplicial volume of non-compact manifolds can be infinite; e.g.,  $\|\mathbb{R}\| = \infty$ . In particular, if  $(W, \partial W)$  is an oriented compact connected manifold with boundary, then in general the simplicial volume of the interior  $W^\circ$  and the relative simplicial volume of  $(W, \partial W)$  (which is always finite) do not coincide; the finiteness of  $\|W^\circ\|$  can be characterised in terms of  $\ell^1$ -homology [24].
- However, the simplicial volume of non-compact manifolds vanishes for many interesting manifolds [13, p. 58f][20], and it does not behave well with respect to taking products: for instance,  $\|\mathbb{R}\| = \infty$ , but  $\|\mathbb{R} \times \mathbb{R}\| = 0$  [13, p. 8f][24].

(2) *Geometric version* [13, Section 4.4f][20]: Let  $M$  be an oriented connected Riemannian  $n$ -manifold. Then the **Lipschitz simplicial volume** of  $M$  is defined by

$$\|M\|_{\text{Lip}} := \inf \left\{ \|c\|_1 \mid c \in C_n^{\text{lf}}(M; \mathbb{R}) \text{ is a fundamental cycle of } M \right. \\ \left. \text{with } \text{Lip}(c) < \infty \right\} \in [0, \infty],$$

where  $\text{Lip}(c)$  is the supremum of the Lipschitz constants of all singular simplices occurring in  $c$ .



- Lipschitz simplicial volume is functorial with respect to proper Lipschitz maps between Riemannian manifolds of the same dimension (so, one application of the Lipschitz simplicial volume are degree theorems for proper Lipschitz maps between non-compact manifolds [20]).
- The Lipschitz simplicial volume is (up to a dimension constant) a lower bound for the minimal volume [13].
- The Lipschitz simplicial volume of locally symmetric spaces of finite volume and non-compact type is non-zero [20].
- The Lipschitz simplicial volume of Hilbert modular varieties coincides with the simplicial volume [21].
- The Lipschitz simplicial volume behaves better than the topologically defined simplicial volume with respect to taking products [20], and there is a version of the proportionality principle for the Lipschitz simplicial volume [13, p. 80][20]

**6.3. Simplicial volume of non-orientable manifolds.** The simplicial volume of a connected non-orientable manifold  $M$  is defined as  $1/2 \cdot \|\overline{M}\|$ , where  $\overline{M}$  is the orientation double covering of  $M$ ; in view of the multiplicativity of the simplicial volume (of oriented manifolds) with respect to finite coverings, this definition is reasonable.

## 7. APPLICATIONS

Typically, simplicial volume is used as a tool to establish topological rigidity properties of the Riemannian volume or non-vanishing of the minimal volume of certain manifolds. Two prominent examples are Gromov's proof of Mostow rigidity and degree theorems.

**7.1. Mostow rigidity.** Hyperbolic manifolds are completely determined by their homotopy type, i.e., hyperbolic manifolds are *rigid* in the following sense [30]:

**Theorem 7.1** (Mostow rigidity). *Let  $M$  and  $N$  be oriented closed connected hyperbolic manifolds of the same dimension  $\geq 3$ . If  $f: M \rightarrow N$  is a homotopy equivalence, then there is an isometry  $M \rightarrow N$  homotopic to  $f$ . In particular, any oriented closed connected smooth manifold of dimension at least 3 can admit at most one hyperbolic structure.*

One decisive step in Gromov's proof [31][2, Chapter C][34, Chapter 11] of Mostow rigidity is to show that if two oriented closed connected hyperbolic manifolds are homotopy equivalent, they have to have the same volume; Gromov introduced simplicial volume in this context as it provides an elegant way of establishing this fact (using the relation to the hyperbolic volume (Theorem 3.3) and the functoriality of the simplicial volume (Corollary 2.2)).

**7.2. Degree theorems.** A degree theorem is a theorem of the following form:

**Metatheorem 7.2.** (Degree theorem) Let  $D$  and  $T$  be certain suitable classes of Riemannian manifolds of the same dimension – the domain manifolds and the target

manifolds. Then there is a constant  $c \in \mathbb{R}_{>0}$  with the following property: For all  $M \in D$ , all  $N \in T$  and all continuous maps  $f: M \rightarrow N$  we have

$$|\deg f| \leq c \cdot \frac{\text{vol}(M)}{\text{vol}(N)}.$$

The art is to find suitable classes of domain and target manifolds. The functoriality of simplicial volume (Corollary 2.2) and the relationship between the Riemannian volume of closed hyperbolic manifolds and the simplicial volume (Theorem 3.3) combine to give a degree theorem for hyperbolic manifolds:

**Theorem 7.3** (Degree theorem for hyperbolic manifolds). *Let  $M$  and  $N$  be oriented closed connected hyperbolic manifolds of the same dimension. Then*

$$|\deg f| \leq \frac{\text{vol}(M)}{\text{vol}(N)}$$

*holds for any continuous map  $f: M \rightarrow N$ .*

In a similar way, non-vanishing results for simplicial volume together with Gromov's estimate of the minimal volume in terms of the simplicial volume give rise to more general degree theorems [13, Section 0.5][19], and parallel results for the Lipschitz simplicial volume of non-compact Riemannian manifolds lead to degree theorems for proper Lipschitz maps between certain non-compact manifolds [20].

**7.3. Dehn fillings and Riemannian volume.** Using the relation between the simplicial volume of hyperbolic manifolds and the hyperbolic volume (a generalised version that also covers non-compact manifolds and manifolds with boundary), Thurston proved that [hyperbolic Dehn fillings](#) of complete hyperbolic 3-manifolds of finite volume decrease the Riemannian volume [35, Theorem 6.5.6].

**7.4. Recognising graph manifolds.** Simplicial volume recognises [graph manifolds](#):

- The gluing formula for gluings along tori [13][18], the fact that the simplicial volume of hyperbolic manifolds is proportional to the Riemannian volume, and the fact that the simplicial volume of [Seifert 3-manifolds](#) is zero show that the simplicial volume of a 3-manifold (possibly with boundary) that has a geometric decomposition is proportional to the sum of the volumes of the hyperbolic pieces in such a decomposition. Therefore, a 3-manifold that has a geometric decomposition is a graph manifold if and only if it has zero simplicial volume. (Together with Perelman's proof of geometrisation for 3-manifolds this means that a 3-manifold is a graph manifold if and only if it has zero simplicial volume.)
- In a similar spirit, one can show the following: Let  $M$  be a [Haken 3-manifold](#) whose boundary is a union of tori such that any manifold obtained from  $M$  by Dehn fillings has simplicial volume equal to zero. Then  $M$  is a graph manifold [5, Proposition 10.17][6, Proposition 9.36]. This result is used in an alternative proof [3] of the last step in Perelman's proof of the Geometrisation Conjecture for aspherical 3-manifolds.

**7.5. A conjecture on the simplicial volume of knot complements.** H. Murakami and J. Murakami conjectured, similar to the volume conjecture for knots, that the simplicial volume of a [knot complement](#) should be related to the asymptotic growth rate of the coloured Jones polynomials of the knot in question [32, Conjecture 5.1]; if this conjecture holds, then the (finite type) [Vassiliev invariants](#) are able to detect triviality of a knot [32].

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CLARA LÖH

NWF I – MATHEMATIK

UNIVERSITÄT REGENSBURG

93040 REGENSBURG, GERMANY

*E-mail address:* [clara.loeh@mathematik.uni-regensburg.de](mailto:clara.loeh@mathematik.uni-regensburg.de)

*Web address:* <http://www.mathematik.uni-regensburg.de/loeh>