Bulletin of the Manifold Atlas (2011)

Super manifolds: an incomplete survey*

HENNING HOHNHOLD, STEPHAN STOLZ AND PETER TEICHNER

Abstract. We present an incomplete survey on some basic notions of super manifolds which may serve as a short introduction to this subject.

58A50

1. Introduction

We present an incomplete survey on some basic notions of super manifolds which may serve as a short introduction to this subject. Almost all the material is taken from the beautiful survey article on super manifolds [3]. Standard references also include [5], [2], [6] or [8]. The material below is a prerequisite to our papers [4] and [7].

2. Super Algebra

Let us begin by explaining briefly what *super* means in an algebraic context, working with the ground field of real numbers. The monoidal category of *super vector spaces*, with tensor products, is by definition the same as the monoidal category of $\mathbb{Z}/2$-graded vector spaces, with the graded tensor product. As a consequence, a super algebra is simply a monoidal object in this category and is hence the same thing as a $\mathbb{Z}/2$-graded algebra. For example, the endomorphism ring $\mathrm{End}(V)$ of a super vector space V inherits a natural $\mathbb{Z}/2$-grading from that of V. The distinction between these notions only arises from the choice of symmetry operators

$$\sigma = \sigma_{V,W} : V \otimes W \xrightarrow{\cong} W \otimes V.$$

There are two standard choices, yielding two very different *symmetric* monoidal categories. For super vector spaces one has

$$\sigma(v \otimes w) = (-1)^{|v|\cdot|w|} w \otimes v,$$

where $|v|$ is the $\mathbb{Z}/2$-degree of a homogenous vector $v \in V$. For $\mathbb{Z}/2$-graded vector spaces the signs would be omitted. This basic difference is sometimes summarized as the

• **Sign rule:** Commuting two odd quantities yields a sign -1.

As a consequence, a super algebra is *commutative* if for all homogenenous $a, b \in A$ we have

$$ab = (-1)^{|a||b|} ba,$$

a very different notion than a commutative $\mathbb{Z}/2$-graded algebra. The standard examples of commutative super algebras are the exterior algebras $\Lambda^*(\mathbb{R}^q)$. As we shall

*Atlas page: www.map.mpim-bonn.mpg.de/Super_manifolds:_an_incomplete_survey

Keywords: super manifolds, super Lie groups, super vector bundles, super algebra

Accepted: 2nd December 2010

see, the generators of $\Lambda^*(\mathbb{R}^q)$ yield the so-called odd coordinates on super manifolds; these anti-commute and hence are useful when trying to describe physical systems involving Fermions. Super algebras also arise naturally in algebraic topology: for every space X, the cohomology ring $H^*(X;\mathbb{R})$ is a commutative super algebra.

Let A be a commutative super algebra. The *derivations* of A are endomorphisms $D \in \mathrm{End}(A)$ satisfying the Leibniz rule:[1]

$$D(a \cdot b) = Da \cdot b + (-1)^{|D||a|} a \cdot Db.$$

$\mathrm{Der}A$ is a super Lie algebra with respect to the bracket operation

$$[D, E] := DE - (-1)^{|D||E|} ED$$

This means that the following axioms are satisfied for $L = \mathrm{Der}A$.

Definition 2.1. A *super Lie algebra* is a super vector space L together with a *Lie bracket* $[\cdot,\cdot] : L \otimes L \to L$ that is skew symmetric

$$[D, E] + (-1)^{|D||E|}[E, D] = 0$$

and satisfies the *Jacobi identity*

$$[D, [E, F]] + (-1)^{|D|(|E|+|F|)}[E, [F, D]] + (-1)^{|F|(|D|+|E|)}[F, [D, E]] = 0.$$

Note that we cyclically permuted the 3 symbols and put down the signs according to the above sign rule.

3. Super Manifolds

We will define super manifolds as ringed spaces following [3]. By a morphism we will always mean a map of ringed spaces. The local model for a super manifold of dimension $p|q$ is $\mathbb{R}^p$ equipped with the sheaf $\mathcal{O}_{\mathbb{R}^{p|q}}$ of commutative super $\mathbb{R}$-algebras $U \mapsto C^\infty(U) \otimes \Lambda^*(\mathbb{R}^q)$.

Definition 3.1. A *super manifold* M of dimension $p|q$ is a pair $(|M|, \mathcal{O}_M)$ consisting of a (Hausdorff and second countable) topological space $|M|$ together with a sheaf of commutative super $\mathbb{R}$-algebras $\mathcal{O}_M$ that is locally isomorphic to $(\mathbb{R}^p, \mathcal{O}_{\mathbb{R}^{p|q}})$. A morphism $f = (|f|, F)$ between super manifolds M, N is defined to be a continuous map $|f| : |M| \to |N|$, together with a map F of sheaves covering $|f|$. More precisely, for every open subset $U \subseteq |N|$ there are algebra maps

$$F(U) : \mathcal{O}_N(U) \to \mathcal{O}_M(|f|^{-1}(U))$$

that are compatible with the restriction maps of the two sheaves. In the future we shall write f^* for F and we denote this category of super manifolds by SM.

To every super manifold M there is an associated *reduced manifold*

$$M^{red} := (|M|, \mathcal{O}_M/\mathrm{Nil})$$

obtained by dividing out the ideal of nilpotent functions. By construction, this gives a smooth manifold structure on the underlying topological space $|M|$ and there is an inclusion of super manifolds $M^{red} \hookrightarrow M$. Note that the sheaf of ideals $\mathrm{Nil} \subset \mathcal{O}_M$

[1] Whenever we write formulas involving the degree $|.|$ of certain elements, we implicitly assume that these elements are homogenous.

is generated by odd functions. Other geometric super objects can be defined in a similar way. For example, replacing $\mathbb{R}$ by $\mathbb{C}$ and C^∞ by analytic functions one obtains *complex (analytic) super manifolds*. There is also an important notion of *cs manifolds*. These are spaces equipped with sheaves of commutative super $\mathbb{C}$-algebras that locally look like $\mathcal{O}_{\mathbb{R}^{p|q}} \otimes \mathbb{C}$. One relevance of *cs* manifolds is that they appear naturally as the smooth super manifolds underlying complex analytic super manifolds. In our work, cs manifolds are essential to define the notion of a *unitary* field theory but this is not relevant for the current discussion.

Definition 3.2. Let E be a real vector bundle of fiber dimension q over the ordinary manifold X^p and $\Lambda^*(E^*)$ the associated algebra bundle of alternating multilinear forms on E. Then its sheaf of sections gives a super manifold $(X, \Gamma(\Lambda^* E^*))$ of dimension $p|q$, denoted by ΠE. In the current *smooth* setting, Marjorie Batchelor proved in [1] that every super manifold is isomorphic to one of this type (this is not true for analytic super manifolds). More precisely, let BunMan denote the category of real vector bundles over smooth manifolds, and for $M \in \mathsf{SM}$, consider the vector bundle $J(M)$ over M^{red} with sheaf of sections $\mathrm{Nil}/\mathrm{Nil}^2$. Then the functors

$$\Pi : \mathsf{BunMan} \to \mathsf{SM} \quad \text{and} \quad J : \mathsf{SM} \to \mathsf{BunMan}$$

come equipped with natural isomorphisms $J \circ \Pi(E) \cong E$ but there are only *non-natural* isomorphisms $\Pi \circ J(M) \cong M$, coming from a choice of a partition of unity. In other words, these functors induce a bijection on isomorphism classes of objects and inclusions on morphisms but they are not equivalences of categories because there are many more morphims in SM than the linear bundle maps coming from BunMan.

The following proposition gives two extremely useful ways of looking at morphisms between super manifolds. We shall use the notation $C^\infty(M) := \mathcal{O}_M(M)$ for the algebra of (global) functions on a super manifold M.

Proposition 3.3. *For $S, M \in \mathsf{SM}$, the functor C^∞ induces natural bijections*

$$\mathsf{SM}(S, M) \cong \mathsf{Alg}(C^\infty(M), C^\infty(S)).$$

If $M \subseteq \mathbb{R}^{p|q}$ is an open super submanifold (a domain), $\mathsf{SM}(S, M)$ is in bijective correspondence with those $(f_1, ..., f_p, \eta_1, ..., \eta_q)$ in $(C^\infty(S)^{ev})^p \times (C^\infty(S)^{odd})^q$ that satisfy

$$(|f_1|(s), ..., |f_p|(s)) \in |M| \subseteq \mathbb{R}^p \text{ for all } s \in |S|.$$

The f_i, η_j are called the coordinates of $\phi \in \mathsf{SM}(S, M)$ defined by

$$f_i = \phi^*(x_i) \quad \text{and} \quad \eta_j = \phi^*(\theta_j),$$

where $x_1, ..., x_p, \theta_q, ..., \theta_p$ are coordinates on $M \subseteq \mathbb{R}^{p|q}$. Moreover, by the first part we see that $f_i \in C^\infty(S)^{ev} = \mathsf{SM}(S, \mathbb{R})$ and hence $|f_i| \in \mathsf{Man}(|S|, \mathbb{R})$.

The proof of the first part is based on the existence of partitions of unity for super manifolds, so it is false in analytic settings. The second part always holds and is proved in [5].

4. The Functor of Points

Since sheaves are generally difficult to work with, one often thinks of super manifolds in terms of their S-points, i.e. instead of M itself one considers the morphism sets $\mathsf{SM}(S,M)$, where S varies over all super manifolds S. More formally, embed the category SM of super manifolds in the category of contravariant functors from SM to Set by

$$Y : \mathsf{SM} \to \mathrm{Fun}(\mathsf{SM}^{op}, \mathsf{Set}), \quad Y(M) = (\ S \mapsto \mathsf{SM}(S,M)\ ).$$

This Yoneda embedding is fully faithful and identifies SM with the the category of *representable* functors, defined to be those in the image of Y. We will sometimes refer to an arbitrary functor $F : \mathsf{SM}^{op} \to \mathsf{Set}$ as a *generalized super manifold.* Note that Proposition 3.3 makes it easy to describe the morphism sets $\mathsf{SM}(S,M)$. We'd also like to point out that the functor of points approach is closely related to computations involving additional odd quantities (the odd coordinates of S as opposed to those of M) in many physics papers.

5. Super Lie Groups

These are simply group objects in SM. According to the functor of points approach, such a group object in SM can be described by giving a functor $G : \mathsf{SM}^{op} \to \mathsf{Group}$ such that the composition with the forgetful functor $\mathsf{Group} \to \mathsf{Set}$ is representable.

Example 5.1. The simplest super Lie group is the additive group structure on $\mathbb{R}^{p|q}$. It is given by the following composition law on $\mathsf{SM}(S,\mathbb{R}^{p|q})$, obviously natural in S:

$$(f_1, ..., \eta_q) \times (h_1, ..., \psi_q) \mapsto (f_1 + h_1, ..., \eta_q + \psi_q).$$

The *super general linear group* $GL(p|q)$ is defined by

$$GL(p|q)(S) := \mathrm{Aut}_{\mathcal{O}_S}(\mathcal{O}_S^{p|q}) \cong \mathrm{Aut}_{C^\infty(S)}(C^\infty(S)^{p|q}),$$

where $A^{p|q}$ denotes the A-module freely generated by p even and q odd generators. We need to check that this is representable. We claim that $GL(p|q)(\_)$ is represented by the open super submanifold $G \subset \mathbb{R}^{p^2+q^2|2pq}$ characterized by

$$|G| = \{\ x \in \mathbb{R}^{p^2+q^2} \mid x \in GL_p \times GL_q\ \}.$$

This follows directly from proposition 3.3 using that a map between super algebras is invertible if and only if it is invertible modulo nilpotent elements.

6. Super Vector Bundles

A (super) vector bundle over a super manifold M is a locally free sheaf $\mathcal{E}$ of $\mathcal{O}_M$-modules of dimension $p|q$. The most basic example of a super vector bundle is the *tangent bundle* of a super manifold $M^{p|q}$. It is the sheaf of $\mathcal{O}_M$-modules $\mathcal{T}M$ defined by

$$\mathcal{T}M(U) := \mathrm{Der}\mathcal{O}_M(U).$$

$\mathcal{T}M$ is locally free of dimension $p|q$: If $x_1, ..., \theta_q$ are local coordinates on M, then a local basis is given by $\partial_{x_1}, ..., \partial_{\theta_q}$. Note that there is also a linear fibre bundle $TM \to M$ with structure group $GL(p|q)$, where TM is a super manifold of dimension

$2p|2q$. More generally, any vector bundle $\mathcal{E}$ over M has a *total space* $E \in \mathsf{SM}$ that comes with a projection map $\pi : E \to M$. It can be most easily described in terms of its S-points

$$E(S) = \{(f, g) \mid f \in \mathsf{SM}(S, M), g \in f^*(\mathcal{E}^{ev}(M))\}.$$

So g is an even global section of the pullback bundle on S and the projection π comes from forgetting this datum. To prove that this functor $\mathsf{SM}^{op} \to \mathsf{Set}$ is representable one uses the local triviality of $\mathcal{E}$ and Proposition 3.3. It follows by construction that the typical fibre of the projection π is $\mathbb{R}^{p|q}$ and the structure group is $GL(p|q)$.

There is an important operation of *parity reversal* on the category of vector bundles over M. It is an involution

$$\Pi : \mathrm{Vect}_M \to \mathrm{Vect}_M$$

that takes a vector bundle E with grading involution α to $(E, -\alpha)$. This means that even and odd parts are exchanged. To define Π on morphisms it is easiest to give it as $\Pi(E) = \epsilon_{0|1} \otimes E$, where $\epsilon_{0|1}$ is the trivial bundle of dimension $0|1$ (aka the constant sheaf of free $\mathcal{O}_M$-modules).

One can define the super Lie algebra $\mathfrak{g}$ of a super Lie group G as follows. A vector field $\xi \in \mathcal{T}G$ is called *left-invariant* if ξ is related to itself under the left-translation by all $f : S \to G$:

$$S \times G \xrightarrow{f\times \mathrm{id}} G \times G \xrightarrow{\mu} G.$$

Here we interpret ξ as a vertical vector field on $S \times G$ in the obvious way. The super Lie algebra $\mathfrak{g}$ consists of all left-invariant vector fields on G. Pulling back via the unit $e : \mathrm{pt} \to G$ defines an isomorphism $\mathfrak{g} \cong T_eG$, in particular, the vector space dimension of $\mathfrak{g}$ is $p|q$.

References

[1] M. Batchelor, *The structure of supermanifolds*, Trans. Amer. Math. Soc. **253** (1979), 329-338. MR536951 Zbl0413.58002

[2] Bernstein, *Algebra and analysis with anticommuting variable*, Reidel 1987.

[3] P. Deligne and J. W. Morgan, *Notes on supersymmetry (following Joseph Bernstein)*, Quantum fields and strings: a course for mathematicians, Vol. 1, 2 (Princeton, NJ, 1996/1997), 41-97, Amer. Math. Soc., Providence, RI, (1999), 41-97. MR1701597 Zbl1170.58302

[4] H. Hohnhold, M. Kreck, S. Stolz and P. Teichner, *Differential Forms and 0-dimensional Super Symmetric Field Theories*, to appear in Quantum Topology, Journal of the European Math. Soc. (2010).

[5] D. A. Leites, *Introduction to the theory of supermanifolds*, Russian Math. Surveys 35 No 1 (1980), 3-57. MR0565567 Zbl0439.58007

[6] Y. I. Manin, *Gauge field theory and complex geometry*, Springer-Verlag, Berlin, 1988. MR954833 Zbl0884.53002

[7] S. Stolz and P. Teichner, *Supersymmetric Euclidean field theories and generalized cohomology*, Survey (2008). Available at http://math.berkeley.edu/˜teichner/Papers/Survey.pdf.

[8] T. Voronov, *Geometric integration theory on supermanifolds*, Harwood Academic Publishers, Chur, 1991. MR1202882 Zbl0839.58014

HENNING HOHNHOLD
STATISTISCHES BUNDESAMT,
GUSTAV-STRESEMANN-RING 11,
65189 WIESBADEN, GERMANY

E-mail address: henninghohnhold@gmail.com

STEPHAN STOLZ
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF NOTRE DAME,
255 HURLEY, NOTRE DAME,
IN 46556, USA

E-mail address: stolz.1@nd.edu
Web address: http://www.nd.edu/ mathwww/faculty/stolz.shtml

PETER TEICHNER
DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF CALIFORNIA, BERKELEY,
CA 94720-3840, USA

AND

MAX PLANCK INSTITUTE FOR MATHEMATICS,
VIVATSGASSE 7,
53111 BONN, GERMANY

E-mail address: teichner@mac.com
Web address: http://web.me.com/teichner/Math