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Hirzebruch surfaces*

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Abstract. We recall the construction and elementary properties of the manifolds now called Hirzebruch surfaces: In a 1951 publication Hirzebruch showed these smooth manifolds admit infinitely many pairwise non-holomorphic complex structures.

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1. Introduction

Hirzebruch surfaces were introduced by Hirzebruch (without that name) in his thesis [2]. They are algebraic surfaces over the complex numbers. Here we consider them as smooth manifolds. The interest in them comes from Hirzebruch's result that as complex manifolds they are pairwise distinct whereas there are only two diffeomorphism types.

2. Construction and examples

Recall that if E is a complex vector bundle over X, then taking the fibre-wise projective space yields the associated projective bundle:

$$P(E) := (E - 0)/\sim, \quad v \sim \lambda w \text{ for all } \lambda \in \mathbb{C} - \{0\}.$$

The fibres of $P(E)$ are complex projective spaces $P(E_x)$ and if E is a holomorphic vector bundle over a complex manifold then $P(X)$ is a complex manifold. Moreover, if $\underline{\mathbb{C}} = X \times \mathbb{C}$ denotes the trivial complex line bundle then $P(E \oplus \underline{\mathbb{C}}) \to X$ admits a canonical section

$$s_\infty : X \to P(E \oplus \underline{\mathbb{C}}), \quad x \mapsto [x, (0, 1)]$$

which takes each point of X to the "line at infinity" in $P(E_x \oplus \mathbb{C})$.

We identify $S^1 \subset \mathbb{C}$ with the unit complex numbers and recall that the 3-sphere, $S^3 = \{(z_1, z_2) \,|\, |z_1|^2 + |z_1|^2 = 1\} \subset \mathbb{C}^2$, admits the free S^1 action defined by the equation: $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2)$. The quotient of this action is $S^3/S^1 = \mathbb{C}\mathrm{P}^1$. For any integer $n \in \mathbb{Z}$ define the complex line bundle $L_n \to \mathbb{C}\mathrm{P}^1$ whose total space is the following quotient of $S^3 \times \mathbb{C}$

$$L_n := (S^3 \times \mathbb{C})/\sim_n, \quad (x, z) \sim_n (\lambda x, \lambda^n z) \text{ for all } \lambda \in S^1,$$

and we map $L_n \to \mathbb{C}\mathrm{P}^1$, via $[x, z] \mapsto [x] \in S^3/S^1 = \mathbb{C}\mathrm{P}^1$. For example, L_1 is the complex line bundle associated to the Hopf fibration and L_{-1} is the tautological line bundle.

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Definition 2.1. For $n \in \mathbb{Z}$ define the *Hirzebruch surface* $H_n := P(L_{-n} \oplus \underline{\mathbb{C}})$. It is a complex manifold of complex dimension 2 but we consider it as a smooth manifold of dimension 4.

The Hirzebruch surfaces H_n are S^2-bundles over S^2. Hence they are closed and, by the orientation coming from the complex structure, oriented 4-dimensional manifolds.

3. Invariants

We list some invariants of the manifolds H_n with explanations below: let $P_x \subset H_n$ denote the fibre over $x \in \mathbb{C}\mathrm{P}^1$.

- $\dim_{\mathbb{R}}(H_n) = 4$ and $\dim_{\mathbb{C}}(H_n) = 2$.
- $\pi_j(H_n) \cong \pi_j(S^2) \times \pi_j(S^2)$: in particular $\pi_1(H_n) = 0$.
- $H_j(H_n) \cong \mathbb{Z},\ 0,\ \mathbb{Z}^2\ 0,\ \mathbb{Z};$ for $j = 0, 1, 2, 3, 4$ and $H_j(H_n) = 0$ for $j > 4$.
- $H_2(H_n) = \mathbb{Z}(\tau) \oplus \mathbb{Z}(\nu)$ has basis with $\tau := [s_\infty(\mathbb{C}P^1)]$ and $\nu := [P_x]$.
- With respect to the above basis the intersection form on $H_2(H_n)$ is given by the following matrix: $\begin{pmatrix} -n & 1 \\ 1 & 0 \end{pmatrix}$.
- The Euler characteristic is given by $e(H_n) = 4$.
- The signature vanishes: $\sigma(H_n) = 0$.
- The first Pontrjagin class of H_n is zero: $p_1(H_n) = 0 \in H^4(H_n)$.
- For the complex manifold H_n, the first Chern class $c_1 \in H^2(H_n)$, is given by $c_1(\tau) = -n + 2$ and $c_1(\nu) = 2$.
- The second Stiefel-Whitney class $w_2(H_n) \in H^2(H_n; \mathbb{Z}_2)$ is given by $w_2(\tau) = n \bmod 2$ and $w_2(\nu) = 0$.
- H_n is a spinable if and only if n is even.

3.1. Explanation.

- The computation of the homotopy groups of H_n follows from the homotopy sequence of a fibration and the existence of the section s_∞.
- The homology groups of H_n can be computed by decomposing $H_n = D \cup_{\mathrm{id}} (-D)$ where D is the 2 disc bundle associated to L_{-n} and using the Mayer-Vietoris sequence.
- The computation of the intersection form follows by inspecting the embedded 2-spheres which represent $H_2(H_n)$ and their normal bundles: in particular we apply the fact that the self intersection number of $s_\infty(\mathbb{C}\mathrm{P}^1)$ is the Euler class of L_{-n} [3, Problem 11-C].
- The signature of H_n is zero since the Hirzebruch surfaces are the boundary of the associated D^3-bundle. One can also see this directly from the intersection form.
- The first Pontrjagin class vanishes as its evaluation on the fundamental class of H_n is an oriented bordism invariant [3, Lemma 17.3].

- For the vaules of $c_1(H_n)$ let TM denote the complex tangent bundle of a complex manifold M then $TH_n|_{s_\infty(\mathbb{C}\mathrm{P}^1)} = T\mathbb{C}\mathrm{P}^1 \oplus L_{-n}$ and $TH_n|_{P_x} = TP_x \oplus \underline{\mathbb{C}}$. Moreover we have $P_x \cong \mathbb{C}\mathrm{P}^1$ and $\langle c_1(T\mathbb{C}\mathrm{P}^1), [\mathbb{C}\mathrm{P}^1]\rangle = 2$ and $\langle c_1(L_n), [\mathbb{C}\mathrm{P}^1]\rangle = n$. Both of these equalities are justified by the fact that the first Chern class of a complex line bundle equals the Euler class of the underlying real 2-plane bundle and the fact that the Euler class can be evaluated by counting the oriented intersection points of a transverse section with the zero section. For the bundle L_n we see that there is a well-defined transverse section $[z_0, z_1] \mapsto [z_0, z_1, z_0^n - z_1^n]$ with precisely n positively oriented intersections with the zero section.
- For $w_2(H_n)$ note that in general the Stiefel-Whitney classes are of a complex manifold are the mod 2 reductions of the Chern classes [3, Problem 14-B].

4. Classification

Theorem 4.1 (Hirzebruch [2])**.** *For the smooth manifolds H_n*

$$H_n \cong H_m \Longleftrightarrow n \equiv m \bmod 2,$$

where $\cong$ means diffeomorphic. Moreover as complex manifolds

$$H_n \cong_{\mathbb{C}} H_m \Longleftrightarrow n = m,$$

where $\cong_{\mathbb{C}}$ means complex diffeomorphic.

For the first statement we see that parity of the intersection form implies that if H_n is diffeomorphic to H_m, then $n \equiv m \bmod 2$. On the other hand the smooth Hirzebruch surfaces are the total spaces of the 2-sphere bundle of a 3-dimensional vector bundle over S^2 and these bundles are classified by $\pi_1(SO(3)) \cong \mathbb{Z}/2$ (note that $SO(3)$ is diffeomorphic to $\mathbb{R}\mathrm{P}^3$). Thus there are precisely two diffeomorphism types of Hirzebruch surfaces. By construction $H_0 = S^2 \times S^2$ and by an easy consideration $H_1 = \mathbb{C}\mathrm{P}^2 \sharp (-\mathbb{C}\mathrm{P}^2)$, where $\sharp$ is the connected sum and $-\mathbb{C}\mathrm{P}^2$ is $\mathbb{C}\mathrm{P}^2$ with the opposite orientation.

For more information on Hirzebruch surfaces, in particular why they are pairwise distinct as complex manifolds, see [2].

5. Further remarks

- The Hirzebruch surfaces show that the smooth $S^2 \times S^2$ and $\mathbb{C}\mathrm{P}^2 \sharp (-\mathbb{C}\mathrm{P}^2)$ both admit infinitely many inequivalent complex structures.
- The smooth manifolds $S^2 \times S^2$ and $\mathbb{C}\mathrm{P}^2 \sharp (-\mathbb{C}\mathrm{P}^2)$ are examples of manifolds with isomorphic homotopy groups but distinct homotopy types.
- The Hirzebruch surfaces are the second stage of the so called Bott towers, which are inductively constructed starting from a point as the total space of a projective bundle associated to $L \oplus \underline{\mathbb{C}}$, where L is a line bundle over a lower Bott tower (for more details see [1]). The classification of Bott towers up to homeomorphism or diffeomorphism is an interesting open problem. In particular one can ask whether the integral cohomology ring determines the homeomorphism or diffeomorphism type as it does for Hirzebruch surfaces.

For 3-stage Bott towers Choi, Masuda and Suh [1, Theorem 1.4] prove that the cohomology ring determines the diffeomorphism type.

- The Hirzebruch surfaces give examples where the isotopy classes of certain diffeomorphims do not contain holomorphic maps (in this case because the diffeomorphisms do not preserve the first Chern class). For example, the connected sum of complex conjugation in both factors of $\mathbb{C}\mathrm{P}^2 \sharp (-\mathbb{C}\mathrm{P}^2)$ is not isotopic to a holomorphic map.

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